On birational invariants A and Ω of algebraic plane curves, Ver. 2

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April 11, 2011

Contents

1 main result

Our purpose is to prove the next inequalities:

$$\sigma \leq (A+2)(A+3), \tag{1}$$

and

$$\sigma \leq {A_1}^2 + 3A_1 + 2\overline{g} + 4. \tag{2}$$

Here an invariant A_1 is defined to be $A - \overline{g}$, which satisfies

$$A_1 = \frac{(2Z-D)\cdot Z}{2} - \frac{D\cdot Z}{2} = Z\cdot K_S,$$

where $Z = D + K_S$. By the way,

$$\omega_1 = \omega - \overline{g} = D\cdot K_S;$$

hence, $A_1 - \omega_1 = K_S^2 = 8 - r$.

The next ineuqalities are quadratic estimates of e interms of A and A_1.

Theorem 1. *If $\sigma \geq 4$, then*

$$e \leq \varepsilon_B({A_1}^2 + 3A_1 + 2\overline{g} + 4), \tag{3}$$

where $\varepsilon_B = 1 + \frac{B}{2}$.

Moreover,

Theorem 2. *If $\sigma \geq 4$, then*

$$e \leq \varepsilon_B(A+2)(A+3). \tag{4}$$

Moreover, under the assumption $\sigma \geq 7$, it follows that

Theorem 3.

$$e \leq \varepsilon_B\frac{(\Omega+5)(\Omega+8)}{9}, \tag{5}$$

where an invariant Ω is defined to be $(3D + K_S)\cdot Z$.

Further,

Theorem 4.

$$e \leq \varepsilon_B(\frac{\Omega_1^2}{9} + \Omega_1 + 2\overline{g} + 4), \tag{6}$$

where an invariant Ω_1 is defined to be $\Omega - 2\overline{g}$, which turns out to be $3A_1$.

2 fundamental equalities

If $B \leq 2$ then we have the fundamental equalities:

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $Y = 8\nu_1 + k + \omega_1$,

from which, we get

$$Y = 8\nu_1 + k + \omega_1 = 8\nu_1 + k + A_1 + r - 8 = 8\overline{\nu_1} + k + r + A_1,$$

where $\overline{\nu_1} = \nu_1 - 1$. Thus introduce invariants $\overline{\nu_j}$ and $\overline{Y}$ by $\overline{\nu_j} = \nu_j - 1$ and $\overline{Y} = \sum_{j=1}^{r} \overline{\nu_j}$, respectively.

Then $\overline{Y} = Y - r$ and

$$\overline{Y} = 8\overline{\nu_1} + k + A_1.$$

Moreover, introduce an invariant $\overline{X}$ by

$$\overline{X} = \sum_{j=1}^{r} \overline{\nu_j}^2 = X - 2Y + r,$$

which satisfies that

$$\overline{X} = 8\overline{\nu_1}^2 + 2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g}.$$

However, if $B > 2$, we have fundamental equalities :

- $\overline{Y} = B_2\sigma + 8\overline{\nu_1} + k + A_1,$
- $\overline{X} = B_2\sigma(\sigma - 2) + 8\overline{\nu_1}^2 + 2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g},$

where $B_2 = B - 2$ for $B \geq 2$. Moreover, if $B \leq 2$, we put $B_2 = 0$.
Defining an invariant $\mathcal{Z}^*$ to be $\overline{\nu_1}\overline{Y} - \overline{X}$, we obtain

$$\mathcal{Z}^* = \sum_{j=2}^{\nu_1 - 1} (\nu_1 - j)(j - 1)t_j.$$

Note:

$$\mathcal{Z}^* = (\nu_1 - 2)x_1 + 2(\nu_1 - 3)x_2 + 3(\nu_1 - 4)x_3 + \ldots,$$

where $x_1 = t_2 + t_{\nu_1 - 1}, x_2 = t_3 + t_{\nu_1 - 2}, x_3 = t_4 + t_{\nu_1 - 3},$.

Moreover, by

$$0 \leq \mathcal{Z}^* = B_2\sigma(2 + \overline{\nu_1} - \sigma) - \overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g}.$$

we get

$$B_2\sigma(\sigma - 2 - \overline{\nu_1}) \leq -\overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g}.$$

Proposition 1. *If $B \geq 3$, then*

$$\sigma(\sigma - 2 - \overline{\nu_1}) \leq \nu_1 A_1 + 2\overline{g} - \overline{\nu_1} k - \tilde{k}. \tag{7}$$

Introducing an invariant λ^* by $\lambda^* = k - A_1$, we obtain

$$0 \leq \mathcal{Z}^* = B_2\sigma(2 + \overline{\nu_1} - \sigma) - \lambda^* \overline{\nu_1} + A_1 + 2\overline{g} - \tilde{k}. \tag{8}$$

Hence,

$$B_2\sigma(-2 - \overline{\nu_1} + \sigma) + \lambda^* \overline{\nu_1} \leq A_1 + 2\overline{g} - \tilde{k}. \tag{9}$$

Proposition 2. *If $B \leq 2$, then*

$$\lambda^* \overline{\nu_1} \leq A_1 + 2\overline{g} - \tilde{k}. \tag{10}$$

2.1 Hartshorne's Lemma

Lemma 1 (Hartshorne). *Suppose that (S, D) is minimal.*

(1) If $B = 0$ or 2 then

$$\sigma A_1 + 4\overline{g} = \sigma Z^2 - 2(\sigma - 2)\overline{g} = (2D + \sigma K_S) \cdot Z \geq 2u(\sigma - 2).$$

(2) If $B = 1$ then $2e - 3\sigma = 2u - p$ and either 1) $2e - 3\sigma \geq 0$ or 2) $3\sigma - 2e > 0$. In the first case, $\sigma A_1 + 4\overline{g} \geq (2u - p)(\sigma - 2) \geq 0$ and in the second case,

$$eA_1 + 6\overline{g} = (3D + eK_S) \cdot Z \geq (p - 2u)(u + \nu_1 - 1) > 0.$$

(3) If $B \geq 2$ then $\sigma A_1 + 4\overline{g} \geq 2u(\sigma - 2) + \sigma(\sigma - 2)(B - 2)$.

(4) If $B \geq 3$ then $\sigma A_1 + 4\overline{g} \geq 2u(\sigma - 2) + \sigma(\sigma - 2)$; in particular, $\sigma A_1 + 4\overline{g} = \sigma Z^2 - 2(\sigma - 2)\overline{g} \geq 2u(\sigma - 2) + \sigma(\sigma - 2)$.

Proof.
From $\sigma K_0 + 2C \sim (2e - \sigma(B + 2))F_c$ and $K_0 + C \sim (\sigma - 2)\Delta_\infty(e - \sigma(B + 2))F_c$, it follows that

$$\begin{aligned}(\sigma K_0 + 2C) \cdot (K_0 + C) =& (2e - \sigma(B + 2))(\sigma - 2) \\ &= (e + e - \sigma B - 2\sigma)(\sigma - 2) \\ &= (2u + (B - 2)\sigma)(\sigma - 2) \\ &= 2u(\sigma - 2) + (B - 2)\sigma(\sigma - 2).\end{aligned}$$

If $B = 0$ then

$$e + e - 2\sigma = 2(e - \sigma) = 2u \geq 0.$$

If $B \geq 2$ then

$$e + e - \sigma B - 2\sigma \geq 2u + \sigma(B - 2) \geq 2u \geq 0.$$

However, if $B = 1$ then

$$e + e - \sigma B - 2\sigma = 2e - 3\sigma = 2u - p.$$

Thus,

$$\begin{aligned}
(\sigma K_S + 2D) \cdot Z &= (\sigma K_0 + 2C) \cdot (K_0 + C) + \sum_{j=1}^{\nu_1} (\sigma - 2\nu_j)(\nu_j - 1) \\
&= (\sigma K_0 + 2C) \cdot (K_0 + C) + p\overline{Y} + 2\mathcal{Z}^* \\
&= (2u - p)(\sigma - 2) + p\overline{Y} + 2\mathcal{Z}^* \\
&\geq (2u - p)(\sigma - 2) \geq 0,
\end{aligned}$$

except for the case when $B = 1$ and $2e - 3\sigma = 2u - p < 0$.

Moreover, when $B = 1$ and $2e - 3\sigma = 2u - p < 0$, we obtain

$$\begin{aligned}
(eK_S + 3D) \cdot Z &= (eK_0 + 3C) \cdot (K_0 + C) + \sum_{j=1}^{\nu_1} (e - 3\nu_j)(\nu_j - 1) \\
&= (p - 2u)(u + \nu_1 - 1) + (p + u)\overline{Y} + 3\mathcal{Z}^* \\
&\geq (p - 2u)(u + \nu_1 - 1) > 0.
\end{aligned}$$

By ν_0 we denote $e - \sigma$. Then $2e - 3\sigma = 3\nu_0 - e$. In the last case, $(eK_0 + 3C) \cdot (K_0 + C) = (e - 3\nu_0)(\nu_0 - 1)$ and so

$$(eK_S + 3D) \cdot Z = \sum_{j=0}^{\nu_1} (e - 3\nu_j)(\nu_j - 1). \tag{11}$$

Q.E.D.

3 case in which $B \geq 3$

Proposition 3. *If $\sigma \geq 4$ and $B \geq 3$, then $\sigma \leq A+2$;*

$$\sigma \leq (A+2)(A+3). \tag{12}$$

Proof.

By Lemma 1(4),

$$\sigma A_1 \geq -4\overline{g} + \sigma(\sigma-2) + 2u(\sigma-2),$$

in other words,

$$A_1 \geq \sigma - 2 - \frac{4\overline{g}}{\sigma} + 2u(1-\frac{2}{\sigma}).$$

Hence,

$$A = A_1 + \overline{g} \geq e - 2 + (1-\frac{4}{\sigma})(u+\overline{g}).$$

If $\overline{g} = -1$ and $u = 0$ then

$$A \geq e - 2 + (1-\frac{4}{\sigma})(u+\overline{g}) = e - 3 + \frac{4}{\sigma}.$$

Thus, $A \geq e-2$.

If $u + \overline{g} \geq 0$ then

$$A \geq e - 2 + \frac{\sigma-4}{\sigma}\overline{g} \geq e-2.$$

Therefore,

$$(A+2)(A+3) \geq \sigma(\sigma+1) > e. \tag{13}$$

3.1 an estimate of lower bound of A

Assuming $B \geq 3$, we shall obtain an estimate of lower bound of A.

Proposition 4. *If $B \geq 3$ and $\nu_1 \geq 4$, then $A \geq 7$. Moreover if $A = 7$ then the type is either $[8*24, 3; 4^{12}, 3, 2^{\delta}]$ or $[8*25, 3; 4^{14}]$ where $\delta = 0, 1, 2$.*

Proof. Assume that $A \leq 7$. By the inequality (7),

$$2\nu_1(\nu_1 - 1) \leq \nu_1 A_1 + 2\overline{g}.$$

From $\overline{g} = A - A_1$, it follows that

$$2\nu_1(\nu_1-1) \leq \nu_1 A_1 + 2\overline{g} = \nu_1 A_1 + 2(A - A_1) = (\nu_1 - 2)A_1 + 2A. \tag{14}$$

Therefore, by $A \leq 7$,

$$2\nu_1(\nu_1 - 1) - (\nu_1 - 2)A_1 \leq 2A \leq 14. \tag{15}$$

By $\nu_1 \geq 4$, we get

$$24 - 2A_1 \leq 2A \leq 14.$$

Hence,

$$5 \leq A_1 = A - \overline{g} \leq 7 - \overline{g}. \tag{16}$$

Thus, $\overline{g} \leq 2$ and we have four cases according to the value of $\overline{g}$.

3.1.1 case in which $\overline{g} = 2$

Assume $\overline{g} = 2$. Then $A_1 = 5$ and $A_1 = Z^2 - 2\overline{g} = Z^2 - 4$. Thus, $Z^2 = 9$. By Lemma 1(4), we have

$$\sigma A_1 + 4\overline{g} \geq \sigma(\sigma - 2), \tag{17}$$

and so

$$A_1 + \frac{4\overline{g}}{\sigma} \geq \sigma - 2.$$

Since $A_1 = 5$ and $\overline{g} = 2$, it follows that

$$7 \geq \sigma - \frac{8}{\sigma}.$$

Hence, $\sigma \leq 8$. However, since $\nu_1 \geq 4$, it follows that $\sigma = 8 = 2 \cdot \nu_1$; thus $\sigma = 8, A = 7, A_1 = 5, p = 0, \tilde{k} = 0$ and $k = 2u$.

By the formula (7), we have

$$24 = \sigma(\sigma - 2 - \overline{\nu_1}) \leq -\overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g} = -6u + 4 \cdot 5 + 4 = 24 - 6u.$$

Therefore, $u = 0$. Hence,

- $\overline{Y} = \sigma + 8\overline{\nu_1} + k + A_1 = 37$,
- $\overline{X} = \sigma(\sigma - 2) + 8\overline{\nu_1}^2 + 2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g} = 48 + 72 - 5 - 4 = 111.$

Then

$$\mathcal{Z}^* = \overline{\nu_1}\overline{Y} - \overline{X} = 111 - 111 = 0.$$

Hence, $\nu_1 = \nu_r = 4$ and $37 = \overline{Y} = 3 \cdot r$, which is a contradiction.

3.1.2 case in which $\overline{g} = 1$

Assume $\overline{g} = 1$. Then $A_1 = A - 1 \geq 5$ and $A \geq 6$. Thus, $A = 6$ or 7.

(1) $A = 6$. Then $A_1 = 6 - 1 = 5$ and

$$\frac{4}{\sigma} + 5 \geq \sigma - 2.$$

This implies

$$7 \geq \sigma - \frac{4}{\sigma}.$$

But since $\sigma \geq 8$, it follows that $\sigma - \frac{4}{\sigma} > 7$, which induces a contradiction.

(2) $A = 7$. Then $A_1 = 6$ and

$$8 \geq \sigma - \frac{4}{\sigma}.$$

Hence, $\sigma = 8; k = 0$. It is easy to see that $\overline{Y} = 38$, $\overline{X} = 120 - 2 = 118$. Hence, $\mathcal{Z}^* = 2$. By $\mathcal{Z}^* = 2(t_2 + t_3)$, we have $t_2 + t_3 = 1$. Thus $t_3 = 1$ or 0.

Moreover,

$$38 = \overline{Y} = 3t_4 + 2t_3 + t_2 = 3t_4 + t_3 + 1.$$

Thus $t_2 = 0, t_3 = 1$ and $t_4 = 12$. The type becomes $[8 * 24, 3; 4^{12}, 3]$.

3.1.3 case in which $\overline{g} = 0$

Assume $\overline{g} = 0$. Then $A_1 = A \geq 5$ and $7 \geq A \geq 5$. Thus, $A = 5$ or 6 or 7.

However, when $\sigma = 8$, we have $\nu_1 = 4$ and by the formula (7)

$$24 = \sigma(\sigma - 2 - \overline{\nu_1}) \leq \nu_1 A_1 + 2\overline{g} - \overline{\nu_1}k - \tilde{k} \leq 4A_1. \tag{18}$$

Hence, $A = A_1 \geq 6$.

(1) $A = 7$. Then $A_1 = A = 7$ and

$$\frac{4\overline{g}}{\sigma} + A_1 = 7 \geq \sigma - 2.$$

Hence, $\sigma = 8$ or 9.

(i) $\sigma = 8$. Then by the formula (7), we have

$$24 = \sigma(\sigma - 2 - \overline{\nu_1}) \leq -\overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g} = -6u + 4 \cdot 7 = 28 - 6u.$$

Thus, $u = 0$ and so $k = 0$; hence,

$$\overline{Y} = 8 + 8\overline{\nu_1} + A_1 = 32 + 7 = 39.$$

Moreover,

$$\mathcal{Z}^* = \sigma\overline{\nu_1} - \sigma(\sigma - 2) + \nu_1 A_1 + 2\overline{g} = 24 - 48 + 28 = 4.$$

Hence, by $\mathcal{Z}^* = 2(t_2 + t_3)$, we have $t_2 + t_3 = 2$. Then

$$39 = \overline{Y} = 3t_4 + 2t_3 + t_2 = 3t_4 + t_3 + 2.$$

Thus $t_2 = t_3 = 1$ and $t_4 = 12$. The type becomes $[8 * 24, 3; 4^{12}, 3, 2]$.

(ii) $\sigma = 9$. Then $p = 1$ and by the formula (7), we have

$$36 = \sigma(\sigma - 2 - \overline{\nu_1}) \leq -\overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g} = -3k - \tilde{k} + 28 \leq 28.$$

This is a contradiction.

3.1.4 case in which $\overline{g} = -1$

Assume that $\overline{g} = -1$. Then $A_1 = A + 1 \geq 5$ and $7 \geq A \geq 5$. Thus, $A = 5$ or 6 or 7.

(1) $A = 7$. Then $A_1 = 8$ and

$$8 = A_1 \geq \sigma - 2 + \frac{4}{\sigma}.$$

Hence, $\sigma = 8$ or 9.

(i) $\sigma = 9$. Then $p = 1, \nu_1 = 4$ and by the formula (7), we have

$$36 = \sigma(\sigma - 2 - \overline{\nu_1}) \leq \nu_1 A_1 + 2\overline{g} = 36 - 2 = 34.$$

This is a contradiction.

(ii) $\sigma = 8$. Then $p = 0, \nu_1 = 4$ and by the formula (7), we have

$$24 = \sigma(\sigma - 2 - \overline{\nu_1}) \leq -6u + \nu_1 A_1 + 2\overline{g} = 32 - 2 - 6u.$$

Thus $u = 1$ or 0.

(a) $u = 1$. Then $\mathcal{Z}^* = 0$ and $3r = \overline{Y} = 8 + 8 \times 3 + 8 + 2$. Therefore, $r = 14$. The type becomes $[8 * 25, 3; 4^{14}]$.

(b) $u = 0$. Then $\mathcal{Z}^* = 6$ and $\mathcal{Z}^* = 2(t_2 + t_3)$. Hence,$t_2 + t_3 = 3$. By the way, $\overline{Y} = 8+8\times 3+8 = 40$ and $\overline{Y} = 3t_4 + 2t_3 + t_2 = 3t_4 + t_3 + 3$. Therefore, $37 = 3t_4 + t_3; t_4 = 12, t_3 = 1, t_2 = 2$. The type becomes $[8 * 24, 3; 4^{12}, 3, 2^2]$.

(2) $A = 6$. Then $A_1 = 7$ and

$$7 = A_1 \geq \sigma - 2 + \frac{4}{\sigma}.$$

Hence, $\sigma = 8$.

Then $p = 0, \nu_1 = 4$ and by the formula (7), we have

$$24 = \sigma(\sigma - 2 - \overline{\nu_1}) \leq -6u + \nu_1 A_1 + 2\overline{g} = 28 - 2 - 6u.$$

Thus $u = 0$ and so

$$\mathcal{Z}^* = \sigma\overline{\nu_1} - \sigma(\sigma - 2) + \nu_1 A_1 + 2\overline{g} = 24 - 48 + 28 - 2 = 2.$$

Thus, $2(t_2 + t_3) = 2$; hence, $t_2 + t_3 = 1$. Furthermore,

$$\overline{Y} = 8 + 8 \times 3 + 7 = 3t_4 + 2t_3 + t_2 = 3t_4 + t_3 + 1.$$

(3) $A = 5$. Then $A_1 = 6$ and

$$6 = A_1 \geq \sigma - 2 + \frac{4}{\sigma}.$$

Hence, $\sigma < 8$, which contradicts the hypothesis.

4 proof of the inequality (2)

We shall prove the inequality (2); that is

$$\sigma \leq A_1^2 + 3A_1 + 2\overline{g} + 4. \tag{19}$$

4.1 nonsingular case

Suppose that $r = 0, \sigma \geq 3$. Then

$$A_1 = Z^2 - 2\overline{g} = \tau_2 - \tau_1 = -B' + 8,$$

where $B' = 2\sigma + \tilde{B} \geq 3\sigma > 8$.

Moreover, $2\overline{g} = \tau_1 = (\sigma - 1)(\tilde{B} - 2) = \sigma\tilde{B} - B' + 2$.

It is easy to verify that if $B = 1$ then $B' \geq 3\sigma + 4$. Otherwise $B' \geq 4\sigma$.

Hence,

$$\begin{aligned} {A_1}^2 + 3A_1 + 2\overline{g} + 4 - \sigma &= (-B' + 8)(-B' + 11) + \sigma\tilde{B} - B' + 4 - \sigma \\ &= B'^2 - 20B' + 88 + \sigma\tilde{B} + 4 - \sigma. \end{aligned}$$

If $B = 1$ then then $B' \geq 3\sigma + 4$ and

$$\begin{aligned} B'^2 - 20B' + 88 + \sigma\tilde{B} + 4 - \sigma &\geq (3\sigma + 4)(3\sigma - 16) + 88 + \sigma\tilde{B} + 4 - \sigma \\ &= 9\sigma^2 - 36\sigma + 8 + \sigma\tilde{B} - \sigma \\ &= 9\sigma(\sigma - 4 + \tilde{B}/9) + 8 - \sigma \\ &> 0. \end{aligned}$$

By the same reasoning, if $B \neq 1$ then then $B' \geq 4\sigma$ and

$$B'^2 - 20B' + 88 + \sigma\tilde{B} + 4 - \sigma \geq 4\sigma(4\sigma - 20) + 88 + \sigma\tilde{B} + 4 - \sigma > 0.$$

4.2 case in which $B \geq 3$

By Lemma 1(4), we get $\sigma A_1 + 4\overline{g} \geq \sigma(\sigma - 2)$; thus

$$\sigma \leq A_1 + \frac{4\overline{g}}{\sigma} + 2, \tag{20}$$

and

$$\begin{aligned} {A_1}^2 + 3A_1 + 2\overline{g} + 4 - \sigma &\geq {A_1}^2 + 3A_1 + 2\overline{g} + 4 - (A_1 + \frac{4\overline{g}}{\sigma}) \\ &= {A_1}^2 + 2A_1 + 2(1 - \frac{2}{\sigma})\overline{g}. \end{aligned}$$

If $\overline{g} \geq 1$ then

$$A_1{}^2 + 2A_1 + 2(1 - \frac{2}{\sigma})\overline{g} \geq (A_1 + 1)^2 + (1 - \frac{4}{\sigma}) \geq 0.$$

If $\overline{g} = 0$ then

$$A_1{}^2 + 2A_1 + 2(1 - \frac{2}{\sigma})\overline{g} \geq (A_1 + 1)^2 - 1 \geq 4 - 1 = 3.$$

Assume that $A_1 + 1 = 0$. Then $-1 = A_1 = A - \overline{g} = A$, which contradicts $A \geq 0$. Therefore, $A_1{}^2 + 3A_1 + 2\overline{g} + 4 \geq \sigma$.

If $\overline{g} = -1$ then $A_1 = A + 1 \geq 1$. Hence, $A_1 + 1 \geq 2$ and $(A_1 + 1)^2 \geq 4$.

$$A_1{}^2 + 2A_1 + 2(1 - \frac{2}{\sigma})\overline{g} \geq (A_1 + 1)^2 - 1 \geq -1.$$

4.3 case in which $B \leq 2$

By Proposition 2,

$$\lambda^* \overline{\nu_1} \leq A_1 + 2\overline{g} - \tilde{k}.$$

Thus, we distinguish the following cases:

(1) $\lambda^* \geq 1$ and (2) $\lambda^* \leq 0$.

5 case in which $\lambda^* \geq 1$

(1) $\lambda^* \geq 1$.

5.1 case in which $B = 0$

(e-0) $B = 0$.

Suppose that $\mathcal{Z}^* \geq 1$. i.e., $\mathcal{Z}^* \geq \nu_1 - 2$.

Then

$$\begin{aligned}\nu_1 - 2 &\le \mathcal{Z}^* \\ &= -\lambda^*\overline{\nu_1} + A_1 + 2\overline{g} - \tilde{k} \\ &\le -\overline{\nu_1} + A_1 + 2\overline{g} - \tilde{k}.\end{aligned}$$

Hence,

$$2\nu_1 - 3 = \nu_1 - 2 + \overline{\nu_1} \le A_1 + 2\overline{g} - \tilde{k}.$$

Thus,

$$\sigma = 2\nu_1 + p \le A_1 + 2\overline{g} - \tilde{k} + 3 + p. \tag{21}$$

However, by $\tilde{k} - p \ge 0$, we obtain

$$\begin{aligned}{A_1}^2 + 3A_1 + 2\overline{g} + 4 - \sigma &\ge {A_1}^2 + 3A_1 + 2\overline{g} + 4 - (A_1 + 2\overline{g} - \tilde{k} + 3 + p) \\ &= {A_1}^2 + 2A_1 + \tilde{k} - p + 1 \\ &= (A_1 + 1)^2 + \tilde{k} - p \\ &\ge 0.\end{aligned}$$

Since $e = \sigma + u$, it follows that

$${A_1}^2 + 3A_1 + 2\overline{g} + 4 - e \ge (A_1 + 1)^2 + \tilde{k} - p - u, \tag{22}$$

where $k = 4p + 2u$. Then

$$\tilde{k} - p - u = p(k - 2p) - p - u = 2p^2 - p + 2up - u.$$

Thus, if $p > 0$ then $\tilde{k} - p - u > 0$ and so the inequality:

$$e \le {A_1}^2 + 3A_1 + 2\overline{g} + 4. \tag{23}$$

has been established.

However, if $p = 0$, we shall distinguish the various cases according to $\mathcal{Z}^*$.

5.2 $\mathcal{Z}^* > \nu_1 - 2$

(i) $\mathcal{Z}^* > \nu_1 - 2$. In this case, $\mathcal{Z}^* \ge 2\nu_1 - 4$.

Then since $\overline{\nu_1} \ge 2, p = 0$ and $\mathcal{Z}^* = k - A_1 = 2u - A_1$, it follows that

$$\begin{aligned}2\nu_1 - 4 &\le \mathcal{Z}^* \\ &= -\lambda^*\overline{\nu_1} + A_1 + 2\overline{g} \\ &\le -2\lambda^* + A_1 + 2\overline{g} \\ &= 3A_1 - 4u + 2\overline{g}.\end{aligned}$$

Hence,

$$\sigma - 4 = 2\nu_1 - 4 \leq 3A_1 - 4u + 2\overline{g}.$$

Thus,

$$3A_1 + 2\overline{g} + 4 - u - \sigma \geq 3u.$$

Therefore,

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 - (\sigma + u) \geq A_1{}^2 + 3u \geq 0.$$

Thus , the inequality (23) has been established.

5.3 $\mathcal{Z}^* = \nu_1 - 2$

(ii) $\mathcal{Z}^* = \nu_1 - 2$. In this case, $t_2 + t_{\nu_1 - 1} = 1$.
Thus we have the following two cases to examine, separately.

5.4 case when $t_2 = 1$

(ii-1) $t_2 = 1, t_{\nu_1 - 1} = 0$.
Then the type becomes $[2\nu_1 * (2\nu_1 + u); \nu_1{}^{t_{\nu_1}}, 2]$ and $\widetilde{B} = 2e = 4\nu_1 + 2u$.
Hence,

$$Z^2 = 4(\nu_1 - 1)(2\nu_1 + u - 2) - (\nu_1 - 1)^2 t_{\nu_1} - 1.$$

Moreover,

$$\overline{g} = (2\nu_1 - 1)(2\nu_1 + u - 1) - \frac{\nu_1(\nu_1 - 1)}{2} t_{\nu_1} - 2,$$

and so

$$2\overline{g} = (8 - t_{\nu_1})\nu_1(\nu_1 - 1) + 2u(2\nu_1 - 1) - 2.$$

Then $A_1 = Z^2 - 2\overline{g}$ turns out to be

$$(\nu_1 - 1)(t_{\nu_1} - 8) - 2u + 1.$$

Letting s be $7 - t_{\nu_1}$, we obtain

$$A_1 = -s(\nu_1 - 1) - 2u + 1,$$
$$2\overline{g} = s\nu_1(\nu_1 - 1) + 2u(2\nu_1 - 1) - 2.$$

We have the following three cases to examine.

(aa) $s > 0$. Then

$$\begin{aligned}2\overline{g} &= s\nu_1(\nu_1 - 1) + 2u(2\nu_1 - 1) - 2 \\ &> \nu_1(\nu_1 - 1) + 10u - 2 \\ &> 2\nu_1 + u = \sigma + u.\end{aligned}$$

and so

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 - (\sigma + u) \geq A_1{}^2 + 3A_1 + 4 > 0.$$

(bb) $s = 0$. Then

$$2\overline{g} = 2u(2\nu_1 - 1) - 2, A_1 = -2u + 1.$$

Hence,

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 - (\sigma + u) = 4u^2 - 13u + 4 + 4\nu_1 u - 2\nu_1,$$

that is actually positive, since $\nu_1 > 2$.

(cc) $s < 0$. Then
$A_1 = -s\overline{\nu_1} - 2u + 1 = -2u + 1 - s\overline{\nu_1}$ and

$$\begin{aligned}A_1{}^2 &= 4u^2 + 4u(\overline{\nu_1}s - 1) + (\overline{\nu_1}s - 1)^2, \\ 3A_1 &= -6u - 3(\overline{\nu_1}s - 1), \\ 2\overline{g} + 4 - (\sigma + u) &= s\nu_1(\nu_1 - 1) + 2u(2\nu_1 - 1) - 2\overline{\nu_1} + 2 - u \\ &= s(\overline{\nu_1} + 1)\overline{\nu_1} + 2u(2\overline{\nu_1} + 1) - u - 2\overline{\nu_1} + 2.\end{aligned}$$

Hence, defining b, c by the following

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 - (\sigma + u) = 4u^2 - 2bu + c,$$

we get

$$\begin{aligned} b &= 2(1 - s\overline{\nu_1}) + \frac{7}{2} - (2\overline{\nu_1} + 1) \\ &= -2(1+s)\overline{\nu_1} + \frac{9}{2}. \\ c &= (1 - s\overline{\nu_1})^2 + 3 - 2s\overline{\nu_1} + s\overline{\nu_1}^2 - 2\overline{\nu_1} + 2 \\ &= s(1+s)\overline{\nu_1}^2 - 4s\overline{\nu_1} - 2\overline{\nu_1} + 6. \end{aligned}$$

The function $4u^2 - 2bu + c$ has the minimal value $\frac{4c-b^2}{4}$. Here,

$$\begin{aligned} 4c &= 4s(1+s)\overline{\nu_1}^2 - 16s\overline{\nu_1} - 8\overline{\nu_1} + 24, \\ b^2 &= 4(1+s)^2\overline{\nu_1}^2 - 18(1+s)\overline{\nu_1} + \frac{81}{4}. \end{aligned}$$

Thus

$$4c - b^2 = -4(1+s)\overline{\nu_1}^2 + 2s\overline{\nu_1} + 10\overline{\nu_1} + \frac{15}{4}.$$

Suppose $1 + s < 0$. Then the quadratic function $F(x) = -4(1+s)x^2 + 2sx + 10x + \frac{15}{4}$ has the minimal value at $x_0 = \frac{s+5}{4(1+s)}$.

Since $x_0 < 2$ and

$$F(2) = -16(1+s) + 4s + 20 - \frac{17}{4} = -12s + 4 + \frac{15}{4} > 0,$$

it follows that $F(\overline{\nu_1}) \geq F(2) > 0$.

Therefore, $4c - b^2 > 0$ which implies the result.

Suppose $s = -1$. Then $b = \frac{9}{2}$ and $c = 2\overline{\nu_1} + 4$. Hence, $4u^2 - 2bu + c = 4u^2 + 9bu + 2\overline{\nu_1} + 4 > 0$.

Therefore, the inequality (23) has been established.

5.5 case when $t_{\nu_1 - 1} = 1$

(ii-2) $t_2 = 0, t_{\nu_1-1} = 1$.

Then the type becomes $[2\nu_1 * (2\nu_1 + u); {\nu_1}^{t_{\nu_1}}, \nu_1 - 1]$ and so

$$A_1 = -(7 - t_{\nu_1})\overline{\nu_1} - 2u - 1.$$

Moreover,

$$\overline{g} = (2\nu_1 - 1)(2\nu_1 + u - 1) - \frac{\nu_1(\nu_1 - 1)}{2} t_{\nu_1} - \frac{(\nu_1 - 1)(\nu_1 - 2)}{2} - 1$$

and so

$$\begin{aligned} 2\overline{g} &= 8\nu_1\overline{\nu_1} + 2u(2\overline{\nu_1} + 1) - \overline{\nu_1}\nu_1 t_{\nu_1} - \overline{\nu_1}(\nu_1 - 2) \\ &= (7 - t_{\nu_1})\nu_1\overline{\nu_1} + 2u(2\overline{\nu_1} + 1) + 2\overline{\nu_1}. \end{aligned}$$

Letting $\overline{s}$ be $8 - t_{\nu_1} = s - 1$, we obtain

- $2\overline{g} = \overline{s}\nu_1\overline{\nu_1} + 2u(2\overline{\nu_1} + 1) + 2\overline{\nu_1}$,
- $A_1 = -\overline{s\nu_1} - 2u - 1$.

We have the following three cases to examine.

(aa) $\overline{s} > 0$. Then

$$2\overline{g} = \overline{s}\nu_1\overline{\nu_1} + 2u(2\overline{\nu_1} + 1) + 2\overline{\nu_1} > \sigma + u = 2\nu_1 + u$$

and

$${A_1}^2 + 3A_1 + 2\overline{g} + 4 - (\sigma + u) \geq {A_1}^2 + 3A_1 + 4 > 0.$$

(bb) $\overline{s} = 0$. Then

$$\begin{aligned} A_1 &= -2u - 1. \\ 2\overline{g} &= 2u(\overline{\nu_1} + 1) + 2\overline{\nu_1}. \\ {A_1}^2 + 3A_1 + 2\overline{g} + 4 - (\sigma + u) &= 4u^2 + 4u + 1 - 6u - 3 + 2u(\overline{\nu_1} + 1) \\ &\quad + 2\overline{\nu_1} + 4 - u - 2\nu_1 > 0. \end{aligned}$$

(cc) $\overline{s} < -1$. Then $1 + \overline{s} < 0$. Furthermore,

$$
\begin{aligned}
A_1 &= -2u - 1 - \overline{s\nu_1}, \\
A_1^2 &= 4u^2 + 4u(1+\overline{s\nu_1}) + (1+\overline{s\nu_1})^2, \\
3A_1 &= -6u - 3\overline{s\nu_1} - 3, \\
2\overline{g} + 4 - (\sigma + u) &= 2u(2\overline{\nu_1} + \frac{1}{2}) + 2\overline{\nu_1} + \overline{s}\nu_1\overline{\nu_1} + 4 - 2\overline{\nu_1}.
\end{aligned}
$$

Hence, defining b, c by

$$
{A_1}^2 + 3A_1 + 2\overline{g} + 4 - (\sigma + u) = 4u^2 - 2bu + c,
$$

we get

$$
\begin{aligned}
b &= -2(1+\overline{s})\overline{\nu_1} + \frac{1}{2}, \\
b^2 &= 4(1+\overline{s}^2\overline{\nu_1})^2 - 2(1+\overline{s})\overline{\nu_1}) + \frac{1}{4}, \\
c &= (1+\overline{s\nu_1})^2 - 3(1+\overline{s\nu_1}) + \overline{s}\nu_1\overline{\nu_1} + 2\overline{\nu_1} + 4 - 2\nu_1 \\
&= \overline{s}(\overline{s}+1)\overline{\nu_1}^2, \\
4c &= 4\overline{s}(\overline{s}+1)\overline{\nu_1}^2.
\end{aligned}
$$

The function $4u^2 - 2bu + c$ has the minimal value $\frac{4c-b^2}{4}$. Moreover,

$$
4c - b^2 = -4(\overline{s}+1)\overline{\nu_1}^2 + 2(\overline{s}+1)\overline{\nu_1} - \frac{1}{4}.
$$

Define $F(x)$ to be $-4(\overline{s}+1)x^2 + 2(\overline{s}+1)x - \frac{1}{4}$.

Then the minimal value $\frac{4c-b^2}{4} = \frac{F(\overline{\nu_1})}{4}$.

Since $\overline{s}+1 < 0$, it follows that $F(x)$ has the miniml value at $x = \frac{1}{4}$. By $\overline{\nu_1} \geq 2$, we have $F(\overline{\nu_1}) \geq F(2) = -12(\overline{s}+1) - \frac{1}{4} > 0$. Thus , the inequality (23) has been established.

(dd) $\overline{s} = -1$. Then $b = \frac{1}{2}, c = 0$ and

$$
{A_1}^2 + 3A_1 + 2\overline{g} + 4 - (\sigma + u) = 4u^2 - u \geq 0.
$$

5.6 $\mathcal{Z}^* = 0$

(iii) $\mathcal{Z}^* = 0$. Thus, $\nu_1 = \nu_r$ and $\overline{Y} = r\overline{\nu_1}, \overline{X} = r\overline{\nu_1}^2$. Hence,

- $r\overline{\nu_1} = 8\overline{\nu_1} + k + A_1$,
- $r\overline{\nu_1}^2 = 8\overline{\nu_1}^2 + 2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g}$.

Thus since $s = 8 - r = 8 - t_{\nu_1}$, it follows that

- $0 = s\overline{\nu_1} + k + A_1$,
- $0 = s\overline{\nu_1}^2 + 2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g}$.

Therefore,

- $A_1 = -s\overline{\nu_1} - k$,
- $2\overline{g} = s\overline{\nu_1}(\overline{\nu_1} + 1) + k(1 + 2\overline{\nu_1}) + \tilde{k}$.

Recalling that $B = 0$, we shall prove the iequality (23).
(aa) $s \geq 0$.
(bb) $s = 0$.
In these cases, the proofs are omitted.

(c) $1 + s \leq 0$.
First we notice

$$\begin{aligned} A_1^2 &= k^2 + 2ks\overline{\nu_1} + s^2\overline{\nu_1}^2, \\ 3A_1 &= -3k - 3s\overline{\nu_1}, \\ 2\overline{g} + 4 - e &= k(1 + 2\overline{\nu_1}) + \tilde{k} + s\overline{\nu_1}(\overline{\nu_1} + 1) + 4 - e, \\ &= k(2\overline{\nu_1} + p + \frac{1}{2}) + s\overline{\nu_1}(\overline{\nu_1} + 1) - 2p^2 + 2p - 2\overline{\nu_1} + 2, \end{aligned}$$

where $\tilde{k} = p(k - 2p) = pk - 2p^2, u = 2p - \frac{k}{2}$.
The formula $A_1{}^2 + 3A_1 + 2\overline{g} + 4 - e$ is written as $k^2 - 2bk + c$, where

$$\begin{aligned} b &= -(1 + s)\overline{\nu_1} + \frac{5 - 2p}{4}, \\ c &= s(1 + s)\overline{\nu_1}^2 - 2(1 + s)\overline{\nu_1} - 2p^2 + p + 2. \end{aligned}$$

Furthermore,

$$\begin{aligned}
c &= s(1+s)\overline{\nu_1}^2 - 2(1+s)\overline{\nu_1} - 2p^2 + p + 2. \\
b^2 &= (1+s)^2\overline{\nu_1}^2 + (1+s)\overline{\nu_1}(\frac{2p-5}{2}) + \frac{(5-2p)^2}{16}, \\
c - b^2 &= -(1+s)\overline{\nu_1}^2 - (1+s)(\frac{2p-1}{2})\overline{\nu_1} - 2p^2 + 2p + 2 - \frac{(5-2p)^2}{16} \\
&= -(1+s)\overline{\nu_1}^2 - (1+s)(\frac{2p-1}{2})\overline{\nu_1} + \frac{p(13-9p)}{4} + \frac{7}{16}.
\end{aligned}$$

Define a quadratic function $F(x)$ to be $-(1+s)x^2-(1+s)(\frac{2p-1}{2})x+\frac{p(13-9p)}{4}+\frac{7}{16}$.

The axis of the parabola defined by $y = F(x)$ is less than 1. Hence, $F(\overline{\nu_1}) \geq F(2)$. However, if $p \leq 2$, then

$$\begin{aligned}
F(2) &= -(1+s)(2p+3) + \frac{p(13-9p)}{4} + \frac{7}{16} \\
&\geq 2p + 3 + \frac{p(13-9p)}{4} + \frac{7}{16} > 0.
\end{aligned}$$

Hence, if $p \leq 2$, then the inequality (23) has benn established.

5.7 case in which $p \geq 3$

Since $\lambda^* = k - A_1 \geq 1$ and $A_1 = -s\overline{\nu_1} - k$, it follows that

$$\lambda^* = s\overline{\nu_1} + 2k \geq 1. \tag{24}$$

Then

$$\begin{aligned}
2\overline{g} - e + A_1 &= s\overline{\nu_1}(\overline{\nu_1}+1) + k(1+2\overline{\nu_1}) + \tilde{k} - (p+u+2\nu_1) - s\overline{\nu_1} - k \\
&= \overline{\nu_1}(s\overline{\nu_1} + 2k) + \tilde{k} - (p+u+2\nu_1) \\
&= \overline{\nu_1}\lambda^* + \tilde{k} - (p+u+2\nu_1).
\end{aligned}$$

We have two cases to examine.

1) $\lambda^* \geq 2$.
Then

$$\begin{aligned}
2\overline{g} + 4 - e + A_1 &= \overline{\nu_1}\lambda^* + \tilde{k} - (p + u + 2\nu_1) + 4 \\
&\geq 2\overline{\nu_1} + \tilde{k} - (p + u + 2\nu_1) + 4 \\
&= -2 + \tilde{k} - (p + u) + 4 \\
&= 2 + p(k - 2p) - (p + u) \\
&= 2 + p(2p + 2u) - (p + u) \\
&= 2 + p(2p - 1) + u(2p - 1) > 0.
\end{aligned}$$

Further,

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 - e > A_1{}^2 + 2A_1 + 1 = (A_1 + 1)^2 \geq 0.$$

2) $\lambda^* = 1$. Then $A_1 = k - 1$.

$$\begin{aligned}
2\overline{g} &= s\overline{\nu_1}(\overline{\nu_1} + 1) + k(1 + 2\overline{\nu_1}) + \tilde{k}. \\
A_1 &= -s\overline{\nu_1} - k. \\
2\overline{g} + A_1 &= s\overline{\nu_1}^2 + 2k\overline{\nu_1} + \tilde{k}. \\
&= (s\overline{\nu_1} + 2k)\overline{\nu_1} + \tilde{k}. \\
&= \lambda^*\overline{\nu_1} + \tilde{k}. \\
&= \overline{\nu_1} + \tilde{k}.
\end{aligned}$$

Hence,

$$\begin{aligned}
2\overline{g} + 4 - (\sigma + u) + A_1 &= \overline{\nu_1} + \tilde{k} + 4 - p - 2\nu_1 - u \\
&= \tilde{k} + 3 - p - \nu_1 - u, \\
2A_1 &= A_1 + A_1 \\
&= k - 1 - s\overline{\nu_1} - k \\
&= -s\overline{\nu_1} - 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}2\overline{g} + 4 - (\sigma + u) + 3A_1 &= \overline{\nu_1} + \tilde{k} + 4 - p - 2\nu_1 - u - s\overline{\nu_1} - 1 \\ &= \tilde{k} + 3 - p - 2 - u - (1 + s)\overline{\nu_1} - 1 \\ &\geq \tilde{k} + 1 - p - u \\ &= 2p^2 + 2up + 1 - p - u > 0.\end{aligned}$$

Finally,

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 - e > A_1{}^2 + 2p^2 + 2up + 1 - p - u > 0.$$

6 case in which $B = 1$

(e-1) $B = 1$.

Then $e = \sigma + u + \nu_1$, $\widetilde{B} = 2e - \sigma = \sigma + 2u + 2\nu_1$ and what we shall prove is the following inequality:

$$e = \sigma + u + \nu_1 = p + u + 3\nu_1 \leq \frac{3}{2}(A_1{}^2 + 3A_1 + 2\overline{g} + 4).$$

Hence, by the inequality (21),

$$3(A_1{}^2 + 3A_1 + 2\overline{g} + 4) - 2(\sigma + u + \nu_1) \geq 3(A_1 + 1)^2 + 3\tilde{k} - 2k - 2u.$$

Supposing that $p > 0$, we have

$$\begin{aligned}3\tilde{k} - 2k - 2u &= 3p(k - 2p) - 2k - 2u \\ &= 3p(p + 2u) - 6p - 6u \\ &= 3p^2 - 6p - 6(p - 1)u.\end{aligned}$$

The last term is positive except for $p = 1, 0$.

Suppose that $p = 1$. Then we may assume that $A_1 = -1$.

We have $e = \sigma + u + \nu_1, \widetilde{B} = 2e - \sigma = \sigma + 2u + 2\nu_1 = 1 + 2u + 4\nu_1$.

We shall distinguish the various cases according to $\mathcal{Z}^*$.

(i) $\mathcal{Z}^* > \nu_1 - 2$. In this case, $\mathcal{Z}^* \geq 2\nu_1 - 4$.

Here, we treat in the case when $k > 0$.

$$\begin{aligned} 2\nu_1 - 4 &\leq \mathcal{Z}^* \\ &= -\lambda^* \overline{\nu_1} + A_1 + 2\overline{g} - \tilde{k} \\ &\leq -2\lambda^* + A_1 + 2\overline{g} - \tilde{k} \\ &= 3A_1 - 2k + 2\overline{g} - \tilde{k}. \end{aligned}$$

Hence,

$$\sigma - 4 - p = 2\nu_1 - 4 \leq 3A_1 2\overline{g} - 2k - \tilde{k}.$$

Thus,

$$3A_1 + 2\overline{g} + 4 - (u + \sigma) \geq 2k + \tilde{k} - p - u - 4.$$

If $p > 0$ then

$$2k + \tilde{k} - p - u - 4 = 6p + 4u + \tilde{k} - p - u > 0.$$

If $p = 0$ then

$$2k + \tilde{k} - p - u - 4 = 6p + 4u - u - 4 = 3u - 4.$$

If $u > 1$,then $2k + \tilde{k} - p - u - 4 \geq 0$.

$${A_1}^2 + 3A_1 + 2\overline{g} + 4 - (u + \sigma) \geq {A_1}^2 \geq 0.$$

Thus

$${A_1}^2 + 3A_1 + 2\overline{g} + 4 - \frac{2}{3}e \geq {A_1}^2 + 3A_1 + 2\overline{g} + 4 - (\sigma + u).$$

In the argument, case when $u = 1, p = 0$ is excluded. But in our case $A_1 = -1$. Thus, ${A_1}^2 = 1$.

(ii) $\mathcal{Z}^* = \nu_1 - 2$. In this case, $t_2 + t_{\nu_1 - 1} = 1$.

(ii-1) $p = 1$.
Suppose that $t_2 = 1$. Then $t_{\nu_1 - 1} = 0$. By using the following invariant

$$A_{10} = (\sigma - 2)(\widetilde{B} - 4) - (\sigma - 1)(\widetilde{B} - 2) + 2.$$

we get

$$A_{10} = -2u - 8t_{\nu_1} + 8 - 3 = -2u - 8\overline{\nu_1} - 3,$$

and

$$\begin{aligned} A_1 &= A10 + \overline{\nu_1} t_{\nu_1} + 1 \\ &= -2u - 3 - 8\overline{\nu_1} + \overline{\nu_1} t_{\nu_1} + 1 \\ &= -s\overline{\nu_1} - 2u - 2. \end{aligned}$$

By $A_1 = -1$, we get

$$s\overline{\nu_1} = -2u - 1.$$

Hence, $s < 0$.

Moreover, it is easy to see that

$$2\overline{g} = s\overline{\nu_1}\nu_1 + 6\nu_1 - 2 + 4u\nu_1.$$

Then

$$\begin{aligned} A_1 &= -s\overline{\nu_1} - 2u - 2, \\ {A_1}^2 &= 4u^2 + 4u(s\overline{\nu_1} + 2) + (s\overline{\nu_1} + 2)^2, \\ 3A_1 &= -6u - 3s\overline{\nu_1} - 6, \\ 2\overline{g} + 4 - (u + 2\nu_1 + 1) &= 4u\nu_1 - u + s\overline{\nu_1}\nu_1 + 4\nu_1 - 3. \end{aligned}$$

Thus

$${A_1}^2 + 3A_1 + 2\overline{g} + 4 - (u + 2\nu_1 + 1) = 4u^2 - 2bu + c$$

where

$$\begin{aligned} -2b &= 4(s\overline{\nu_1} + 2) - 6 + 4\nu_1 - 1, \\ 2b &= 4(1 + s)\overline{\nu_1} + 2) - 7 < 0, \\ c &= (s\overline{\nu_1} + 2)^2 - 3s\overline{\nu_1} - 6 + s\overline{\nu_1}\nu_1 + 4\nu_1 - 3 \\ &= s(1 + s)\overline{\nu_1}^2 + 2s\overline{\nu_1} + 3\overline{\nu_1} + 2 > 0. \end{aligned}$$

Consequently,

$${A_1}^2 + 3A_1 + 2\overline{g} + 4 - (u + 2\nu_1 + 1) = 4u^2 - 2bu + c > 0.$$

We skip the case when $t_2 = 0, t_{\nu_1+1} = 1$.

(ii-2) $p = 0$.
Assume that $t_2 = 1, t_{\nu_1+1} = 0$.

$$\begin{aligned}
A_1 &= -2u + 1 - s\overline{\nu_1} \\
{A_1}^2 &= 4u^2 - 4u(1 - s\overline{\nu_1}) + (1 - s\overline{\nu_1})^2 \\
3A_1 &= -6u + 3 - 3s\overline{\nu_1} \\
2\overline{g} + 4 - (u + 2\nu_1 + 1) &= 2u(2\overline{\nu_1} + 1) - u + s\nu_1\overline{\nu_1} - 2\overline{\nu_1}
\end{aligned}$$

Then

$${A_1}^2 + 3A_1 + 2\overline{g} + 4 - (u + 2\nu_1 + 1) = 4u^2 - 2bu + c$$

where

$$\begin{aligned}
-2bu &= -4u(1 - s\overline{\nu_1}) - 6u + 2u(2\overline{\nu_1} + 1) - u, \\
b &= -(1 + s)\overline{\nu_1} + \frac{9}{2}, \\
c &= (1 - s\overline{\nu_1})^2 + 3 - 3s\overline{\nu_1} + s\nu_1\overline{\nu_1} - 2\overline{\nu_1}
\end{aligned}$$

7 case in which $\lambda^* \leq 0$

Assume $k - A_1 = \lambda^* \leq 0$. Then $A_1 \geq k \geq 0$ and applying Lemma of Matsuda and Tanaka to the fundamental equalities :

- $\overline{Y} = 8\overline{\nu_1} + k + A_1$,
- $\overline{X} = 8\overline{\nu_1}^2 + 2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g}$,

we obtain

$$\overline{V} = (k + A_1)^2 - (2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g}) \geq 0.$$

7.1 case in which $k > 0$

Supposing that $k > 0$, we get from

$$(k + A_1)^2 - (2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g}) \geq 0,$$

the next estimate:

$$k + 2A_1 + \frac{A_1^2 + A_1 + 2\overline{g} - \tilde{k}}{k} \geq 2\overline{\nu_1} = \sigma - p - 2. \tag{25}$$

(1) $p > 0$.

Assume $p > 0$. Then $A_1 \geq k \geq 3p \geq 3$ and $pk - \tilde{k} = 2p^2$ and $\frac{pk-\tilde{k}}{k} = \frac{2p^2}{k} \leq \frac{2k}{9}$. Hence,

$$\frac{2k}{9} + 2A_1 + \frac{A_1(A_1+1) + 2\overline{g}}{k} \geq k + 2A_1 + \frac{A_1(A_1+1) + 2\overline{g} - \tilde{k}}{k}.$$

Moreover, from $k \leq A_1$, it follows that

$$\frac{2k}{9} + 2A_1 \leq \frac{21A_1}{9} \leq \frac{10A_1}{3}.$$

Therefore,

$$2 + \frac{10A_1}{3} + \frac{A_1^2 + A_1 + 2\overline{g}}{3} = 2 + \frac{{A_1}^2 + 11A_1 + 2\overline{g}}{3},$$

and we obtain

$$\begin{aligned}
{A_1}^2 + 3A_1 + 2\overline{g} + 4 - \sigma &\geq {A_1}^2 + 3A_1 + 2\overline{g} + 4 - (2 + \frac{{A_1}^2 + 11A_1 + 2\overline{g}}{3}) \\
&\geq {A_1}^2 + 3A_1 + 2\overline{g} + 4 - (2 + \frac{{A_1}^2 + 12A_1 + 2\overline{g}}{3}) \\
&\geq \frac{2{A_1}^2}{3} - A_1 + \frac{4\overline{g}}{3} + 2 \\
&\geq \frac{2{A_1}^2 - 3A_1 + 4\overline{g}}{3} + 2 \\
&> \frac{2{A_1}^2 - 4A_1 + 4\overline{g}}{3} + 2 \\
&\geq \frac{2(A_1 - 1)^2 - 2 - 4}{3} + 2 \\
&\geq 2 - \frac{2}{3}.
\end{aligned}$$

Therefore,

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 \geq \sigma + 2.$$

(2) $p = 0, u > 0$.
Assume $p = 0, u > 0$. Then the formula (25) turns out to be

$$2u + 2A_1 + \frac{A_1^2 + A_1 + 2\overline{g}}{2u} \geq 2\overline{\nu_1} = \sigma - 2. \tag{26}$$

Therefore, since $A_1 \geq 2u \geq 2$, we obtain

$$\begin{aligned} A_1{}^2 + 3A_1 + 2\overline{g} + 4 - \sigma &\geq A_1{}^2 + 3A_1 + 2\overline{g} + 4 - (2 + 2u + 2A_1 + \frac{A_1^2 + A_1 + 2\overline{g}}{2u}) \\ &= 2 - 2u + \frac{2u-1}{2u}(A_1{}^2 + A_1 + 2\overline{g}) \\ &\geq 2 - A_1 + \frac{1}{2}(A_1{}^2 + A_1 + 2\overline{g}) \\ &\geq \frac{1}{2}(A_1{}^2 - A_1) + \overline{g} + 2 \\ &\geq 1 + \overline{g} + 2 \\ &\geq 2. \end{aligned}$$

Thus,

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 \geq \sigma + 2.$$

7.2 estimate of e,(2)

(i) $p > 0$.
Since $\frac{\tilde{k}}{k} = \frac{p(k-2p)}{k} = p - \frac{2p^2}{k}$, it follows that

$$\begin{aligned} k + 2A_1 + \frac{A_1^2 + A_1 + 2\overline{g} - \tilde{k}}{k} &= k + 2A_1 + \frac{A_1^2 + A_1 + 2\overline{g}}{k} - p + \frac{2p^2}{k} \\ &\geq \sigma - p - 2. \end{aligned}$$

Thus,

$$k + 2A_1 + \frac{A_1^2 + A_1 + 2\overline{g}}{k} - p + \frac{2p^2}{k} \geq \sigma - p - 2.$$

Hence,

$$k + 2A_1 + \frac{A_1^2 + A_1 + 2\overline{g}}{k} + \frac{2p^2}{k} + 2 \geq \sigma.$$

$$\begin{aligned}
{A_1}^2 + 3A_1 + 2\overline{g} + 4 - e &\geq {A_1}^2 + 3A_1 + 2\overline{g} + 4 \\
&\quad - (k + 2A_1 + \frac{A_1^2 + A_1 + 2\overline{g}}{k} + \frac{2p^2}{k} + 2) - u \\
&= ({A_1}^2 + A_1 + 2\overline{g})(1 - \frac{1}{k}) + \frac{2p^2}{k} + 2 - u - k.
\end{aligned}$$

But by $A_1 \geq k$, we get

$${A_1}^2 + A_1 + 2\overline{g} \geq k^2 + k + 2\overline{g}.$$

Hence,

$$({A_1}^2 + A_1 + 2\overline{g})(1 - \frac{1}{k}) + \frac{2p^2}{k} + 2 - u - k \geq (k^2 + k + 2\overline{g})(1 - \frac{1}{k}).$$

Therefore,

$$\begin{aligned}
{A_1}^2 + 3A_1 + 2\overline{g} + 4 - e &\geq (k^2 + k + 2\overline{g})(1 - \frac{1}{k}) + \frac{2p^2}{k} + 2 - u - k \\
&= (k^2 + k + 2\overline{g})(1 - \frac{1}{k}) + \frac{2p^2}{k} + 2 - u - k \\
&= (k^2 + k)(1 - \frac{1}{k}) + 2\overline{g}(1 - \frac{1}{k}) + \frac{2p^2}{k} + 2 - u - k \\
&= k^2 - 1 - 2(1 - \frac{1}{k}) + \frac{2p^2}{k} + 2 - u - k \\
&> k^2 - 3 + 2 - u - k \\
&= k^2 - 1 - u - k \\
&> 3k - 1 - u - k > 0.
\end{aligned}$$

(ii) $p = 0, u > 0$.
Then

$$2u + 2A_1 + \frac{A_1^2 + A_1 + 2\overline{g}}{2u} + 2 \geq \sigma.$$

Hence,

$$\begin{aligned}
{A_1}^2 + 3A_1 + 2\overline{g} + 4 - e &\geq {A_1}^2 + 3A_1 + 2\overline{g} + 4 - (3u + 2A_1 + \frac{A_1^2 + A_1 + 2\overline{g}}{2u} + 2) \\
&\geq 2 - 3u + \frac{2u-1}{2u}(A_1^2 + A_1 + 2\overline{g}) \\
&\geq 2 - 3u + \frac{2u-1}{u}(2u^2 + u) - \frac{2u-1}{u} \\
&= 2 - 3u + (2u-1)(2u+1) - 2 + \frac{1}{u} \\
&= 2 - 3u + 4u^2 - 1 - 2 + \frac{1}{u} \\
&> 4u^2 + 2u - 2 + \frac{1}{u} > 0
\end{aligned}$$

7.3 case in which $k = 0$

Suppose that $k = 0$ and $-A_1 = k - A_1 = \lambda^* \leq 0$. Thus $A_1 \geq 0$. Putting $t = t_{\nu_1}$, $\overline{Y}' = \sum_{\nu_j < \nu_1} \overline{\nu_j}$ and $\overline{X}' = \sum_{\nu_j < \nu_1} \overline{\nu_j}^2$, we obtain

- $\overline{Y}' = (8 - t)\overline{\nu_1} + A_1$,
- $\overline{X}' = (8 - t)\overline{\nu_1}^2 - A_1 - 2\overline{g}$.

Suppose that $t \geq 8$. Then

$$0 \leq \overline{X}' - \overline{Y}' + (t - 8)\overline{\nu_1}(\overline{\nu_1} - 1) = -2A_1 - 2\overline{g} \leq 2. \tag{27}$$

7.4 case in which $t > 8$

If $\nu_1 = 2$ then from the formula (27) it follows that $\overline{X}' = \overline{Y}' = 0$ and so $r = 8$ or 9; the type beocmes $[4 * 4; 2^r]$. Then $\kappa[D] < 2$, a contradiction.

If $\nu_1 > 2$ then from the formula (27) it follows that $(t-8)\overline{\nu_1}(\overline{\nu_1}-1) = 2$; hence, $\overline{\nu_1} = 2$ and $t = 9$. Therefore, $\nu_1 = 3$ and $\sigma = 6$. Then $g = 5\cdot 5 - 3\cdot 9 < 0$; a contradiction.

7.5 case in which $t = 8$

If $t = 8$ then

$$0 \le \overline{X}' - \overline{Y}' \le -2A_1 - 2\overline{g} \le 2.$$

If $\overline{X}' = \overline{Y}'$ then either 1) $\nu_1 = \nu_r$ or 2) there exists j such that $\nu_j = 2$ for $\nu_j < \nu_1$.

In the case 1), we get $\overline{X}' = \overline{Y}' = 0$ and $A_1 = 0$. Hence, $\overline{g} = 0$. The type beocmes $[2\nu_1 * 2\nu_1; {\nu_1}^8]$. Then $\kappa[D] < 2$, a contradiction.

In the case 2), we get $\overline{X}' = \overline{Y}' = r - 8$ and the fundamental equations turn out to be

- $r - 8 = \overline{Y}' = A_1$,
- $r - 8 = \overline{X}' = -A_1 - 2\overline{g}$.

Thus, $2(r-8) = 2A_1 = -2\overline{g} \le 2$. Therefore, $r = 9$ and $g = 0$. The type beocmes $[2\nu_1 * 2\nu_1; {\nu_1}^8, 2]$. Then $\kappa[D] < 2$, a contradiction.

7.6 case in which $t < 8$

Then $s = 8 - t > 0$ and

- $\overline{Y}' = s\overline{\nu_1} + A_1 = s(\overline{\nu_1} - 1) + s + A_1$,
- $\overline{X}' = s\overline{\nu_1}^2 - A_1 - 2\overline{g}$.

Since $A_1 \ge 0$, it follows that $\overline{Y}' \ge s\overline{\nu_1}$. On the other hand, by Lemma of Matsuda and Tanaka, we obtain

$$\overline{V} = s(\overline{\nu_1} - 1)^2 + (s + A_1)^2 - \overline{X}' \ge 0.$$

Thus

$$\sigma \le 3 + s + 2A_1 + \frac{{A_1}^2 + A_1 + 2\overline{g}}{s}. \tag{28}$$

If $s = 1$ then the inequality (2) has been established.

7.7 lemma

Lemma 2. *If* $\nu_1 > 2$ *then*

$$s < A_1 + \frac{2(A_1 + \overline{g})}{\nu_1 - 2}. \tag{29}$$

Proof.
Since $\mathcal{Z}^* = \sum_{j=2}^{\nu_1 - 1} (\nu_1 - j)(j-1)t_j \geq (\nu_1 - 2)\varepsilon(t)$, it follows that

$$\mathcal{Z}^* = \nu_1 A_1 + 2\overline{g} \geq (\nu_1 - 2)\varepsilon(t).$$

Hence,

$$\varepsilon(t) \leq \frac{\nu_1 A_1 + 2\overline{g}}{\nu_1 - 2} = A_1 + \frac{2A_1 + 2\overline{g}}{\nu_1 - 2}.$$

By the way,

$$s(\nu_1 - 2) < \overline{Y}' = s\overline{\nu_1} + A_1 \leq (\nu_1 - 2)\varepsilon(t).$$

Hence, $s < \varepsilon(t)$; hence,

$$1 + s \leq \varepsilon(t) \leq A_1 + \frac{2A_1 + 2\overline{g}}{\nu_1 - 2}.$$

Therefore,

$$s \leq A_1 + \frac{2A_1 + 2\overline{g}}{\nu_1 - 2} - 1. \tag{30}$$

7.8 case when $s > 1$

It is easy to see that

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 - (3 + s + 2A_1 + \frac{A_1{}^2 + A_1 + 2\overline{g}}{s})$$
$$= (1 - \frac{1}{s})(A_1{}^2 + A_1 + 2\overline{g} - s).$$

Therefore, if $s > 1$ and $A_1{}^2 + A_1 + 2\overline{g} > s$ then

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 \geq 3 + s + 2A_1 + \frac{A_1{}^2 + A_1 + 2\overline{g}}{s} > \sigma.$$

Thus, assume that

$$A_1{}^2 + A_1 + 2\overline{g} \leq s. \tag{31}$$

If $\nu_1 \geq 4$, then by Lemma 2, we obtain

$$s < A_1 + \frac{2(A_1+\overline{g})}{\nu_1 - 2} \leq 2A_1 + \overline{g} - 1.$$

Thus by (31)

$$A_1{}^2 + A_1 + 2\overline{g} \leq s < 2A_1 + \overline{g} - 2.$$

Hence,

$$A_1{}^2 + A_1 + 2\overline{g} \leq 2A_1 + \overline{g} - 2. \tag{32}$$

Therefore,

$$0 \leq A_1{}^2 - A_1 \leq -2,$$

which is a contradiction.

Finally, suppose that $\nu_1 = 3$. Then $\sigma = 6$. The inequality (2) turns out to be

$$6 = \sigma \leq A_1{}^2 + 3A_1 + 2\overline{g} + 4. \tag{33}$$

If $A_1 \geq 2$ then $\leq A_1{}^2 + 3A_1 \geq 4$ and $2\overline{g} + 4 \geq 2$. Thus $A_1{}^2 + 3A_1 + 2\overline{g} + 4 \geq 6$.

To study the case when A_1 or 0 , we recall the formula (29) and we get

$$2 \leq s \leq A_1 + \frac{2A_1 + 2\overline{g}}{\nu_1 - 2} - 1 = 3A_1 + 2\overline{g} - 1.$$

Therefore, if $A_1 = 0$ then $3 \leq 2\overline{g}$; thus $\overline{g} \geq 2$.

Hence,

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 \geq 8.$$

Moreover, if $A_1 = 1$ then $3 \leq 2\overline{g} + 3$; thus $\overline{g} \geq 0$.

$$A_1{}^2 + 3A_1 + 2\overline{g} + 4 \geq 8 + 2\overline{g} \geq 6.$$

This completes the proof of the inequality (2).

8 proof of the inequality (1)

Here we shall derive the inequality (1) from the inequality (2). Since

$$(A+2)(A+3) - (A_1{}^2 + 3A_1 + 2\overline{g} + 4) = g(2A + 2 - \overline{g}),$$

it suffices to prove $2A + 2 - \overline{g} \geq 0$ provided that $g > 0$.

By Lemma 1(2), either (i) $\sigma A_1 \geq -4\overline{g}$ or (ii) $eA_1 \geq -6\overline{g}$.

In case (i), we get $A \geq (1 - \frac{4}{\sigma})\overline{g}$. Hence, it follows that

$$2A + 2 - \overline{g} \geq (1 - \frac{8}{\sigma})\overline{g} + 2. \tag{34}$$

Since $\overline{g} \geq 0$, it follows that $2A + 2 - \overline{g} \geq 0$ provided that $\sigma \geq 8$. Therefore, for $\sigma \leq 7$, it suffices to prove $\sigma \leq (A+2)(A+3)$.

But for $A \geq 1$, we get $(A+2)(A+3) \geq 12$.

In case (ii), we get $A > (1 - \frac{6}{e})\overline{g}$. Hence,

$$2A + 2 - \overline{g} > (1 - \frac{12}{e})\overline{g} + 2. \tag{35}$$

Since $\overline{g} \geq 0$, it follows that $2A + 2 - \overline{g} \geq 0$ provided that $e \geq 12$. Therefore, for $e \leq 11$, it suffices to prove $\sigma \leq (A+2)(A+3)$.

Thus, since $11 \geq e = \sigma + u + \nu_1 \geq \sigma + 2$, it follows that $9 \geq \sigma$.

But for $A \geq 1$, we obtain $(A+2)(A+3) \geq 12 > 9 \geq \sigma$. This completes the proof of the inequality (1).

8.1 case when $g > 1$

Theorem 5. *If $g > 0$ then*

$$\sigma \leq A^2 + 3A + 4. \tag{36}$$

Proof.

If $g = 1$ then the inequality (2) turns out to be

$$\sigma \leq A^2 + 3A + 4. \tag{37}$$

However, if $g > 1$ then

$$A^2 + 3A + 4 - ({A_1}^2 + 3A_1 + 2\overline{g} + 4) = \overline{g}(A + A_1 + 1).$$

Hence, if $A + A_1 + 1 \geq 0$ then

$$A^2 + 3A + 4 - \sigma \geq A^2 + 3A + 4 - ({A_1}^2 + 3A_1 + 2\overline{g} + 4) = \overline{g}(A + A_1 + 1) \geq 0.$$

Therefore, assuming $A + A_1 + 1 < 0$, we shall prove the inequality (37).

However, $A + A_1 + 1 = 2A_1 + \overline{g} + 1 < 0$ and by Lemma 1, we get either 1) $\sigma A_1 + 4\overline{g} \geq 0$ or 2) $eA_1 + 6\overline{g} \geq 1$; i.e., $A_1 > -\frac{6\overline{g}}{e}$.

In case 1), we have $A_1 \geq \frac{-4\overline{g}}{\sigma}$ and so

$$\frac{-8\overline{g}}{\sigma} \leq 2A_1 < -\overline{g} - 1. \tag{38}$$

Thus $\sigma \leq 7$, for $\overline{g} \geq 0$. However, if $A \geq 1$ then $8 \leq A^2 + 3A + 4$. Hence, we obtain the inequality (37)

In case 2), we have $A_1 > \frac{-6\overline{g}}{e}$ and so

$$\frac{-12\overline{g}}{e} < 2A_1 \leq -\overline{g} - 1 < -\overline{g}. \tag{39}$$

Thus $e \leq 11$. By $11 - \sigma \geq e - \sigma \geq \nu_1 \geq 2$, we have $\sigma \leq 9$. In order to prove the inequality (37), it suffices to assume $A = 1$. Suppose that $\sigma = 9$. Then $e = 11, \nu_1 = 2, u = 0$. But from the formula (39), we get

$$\frac{-12\overline{g}}{e} = \frac{-12\overline{g}}{11} < -\overline{g} - 1.$$

Hence,

$$\overline{g} > 11. \tag{40}$$

However, from

$$\frac{-12\overline{g}}{e} = \frac{-12\overline{g}}{11} < 2A_1 = 2(A - \overline{g}),$$

it follows that

$$-12\overline{g} \leq 22A_1 = 22(A - \overline{g}) = 22(1 - \overline{g}) = 22 - 22\overline{g}.$$

Hence, $10\overline{g} \leq 11$; thus $\overline{g} \leq 2$, which contradicts (40).

This completes the proof of the inequality (37).

9 estimate of σ in terms of Ω

Introduce an invariant Ω_1 to be $\Omega - 2\overline{g}$, which turns out to be $3A_1$.

The inequality (2) implies the next:

Theorem 6.

$$9\sigma \leq {\Omega_1}^2 + 9\Omega_1 + 18\overline{g} + 36. \tag{41}$$

In other words,

$$\sigma \leq \frac{{\Omega_1}^2}{9} + \Omega_1 + 2\overline{g} + 4.$$

We shall verify the next inequality:

Theorem 7.

$$\sigma \leq \frac{(\Omega+5)(\Omega+8)}{9}. \tag{42}$$

First, subtract $\frac{(\Omega+5)(\Omega+8)}{9}$ from $\frac{{\Omega_1}^2}{9} + \Omega_1 + 2\overline{g} + 4$.
Denoting ${\Omega_1}^2 + 9\Omega_1 + 18\overline{g} + 36 - (\Omega+5)(\Omega+8)$ by $\tilde{\delta}$ we get

$$\begin{aligned}\tilde{\delta} &= (\Omega_1 - \Omega)(\Omega_1 + \Omega) + 9(\Omega_1 - \Omega) + 18\overline{g} + 36 - \Omega - 40 \\ &= 4(1-g)(\Omega - \overline{g}) + 36 - 40 - 4\Omega \\ &= -4g(\Omega - \overline{g} + 1).\end{aligned}$$

Then

$$\tilde{\delta} = -\frac{4g}{9}(\Omega - \overline{g} + 1). \tag{43}$$

If $g = 0$ then $\tilde{\delta} = 0$. Hence, we can assume $g > 0$.
If $\Omega - \overline{g} + 1 \geq 0$, then $\tilde{\delta} \leq 0$ and

$$\frac{{\Omega_1}^2}{9} + \Omega_1 + 2\overline{g} + 4 \leq \frac{(\Omega+5)(\Omega+8)}{9}.$$

Therefore,in order to show the inequality (42), it suffices to consider in the case when $\Omega - \overline{g} + 1 < 0$.

Note that $g > 1$, since $\Omega - \overline{g} + 1 < 0$ and $\Omega > 0$.

From Lemma 1(2), we have the following result:

If $B = 1$ then either 1) $2e - 3\sigma \geq 0$ or 2) $3\sigma - 2e > 0$. In the first case, $\sigma\Omega_1 + 12\overline{g} \geq 0$ and in the second case, $e\Omega_1 + 18\overline{g} > 0$.

9.1 case 1)

Therefore, in case 1),

$$\Omega - \overline{g} + 1 = \Omega_1 + \overline{g} + 1 \geq (1 - \frac{12}{\sigma})\overline{g} + 1.$$

Thus, if $\sigma \geq 12$, we have the inequality (42).

However, when $\sigma \leq 11$, the inequality (42) is obvious for $\Omega \geq 4$.

So, we assume that $\Omega \leq 3$.

(i) $\Omega = 3$. Since $\Omega - \omega \geq 1$, it follows that $\omega \leq 2$. From

$$\sigma \leq \omega^2 + 3\omega + 2 \leq 12$$

it follows that $\sigma \leq 12$. Hence, the inequality (42) is verified.

(ii) $\Omega = 2$. Since $\Omega - \omega \geq 1$, it follows that $\omega = 1$. From

$$\sigma \leq \omega^2 + 3\omega + 2 = 6$$

it follows that $\sigma \leq 6$. Hence, the inequality (42) is verified, too.

However, when $e \leq 17$, the inequality (42) is obvious for $\Omega \geq 5$.

9.2 case 2)

From $e\Omega_1 + 18\overline{g} > 0$ and $2e > 3\sigma$ it follows that

$$\Omega - \overline{g} + 1 = \Omega_1 + \overline{g} + 1 \geq (1 - \frac{18}{e})\overline{g} + 1.$$

Thus, if $e \geq 18$, we have the inequality (42).

Hereafter, assume $e \leq 17$. Then from $\sigma + u + \nu_1 = e \leq 17$, it follows that $\sigma \leq 15$. By $\nu_0 = e - \sigma < \frac{e}{3} \leq \frac{17}{3}$, we have $\nu_1 \leq \nu_0 \leq 5$. We shall distinguish the various cases according to the value of ν_1.

(1) $\nu_1 = 5$. Hence, $10 = 2\nu_1 \leq \sigma \leq e - \nu_1 = e - 5 \leq 12$.

We consider the plane type $[e; 5^{t_5}, 4^{t_4}, 3^{t_3}, 2^{t_2}]$. Then it is easy to compute Ω and g:

- $\Omega = (e-3)(e-9) + t_2 - 3t_4 - 8t_5$,
- $\overline{g} = \dfrac{e(e-3)}{2} - t_2 - 3t_3 - 6t_4 - 10t_5$.

From these, we obtain

$$5\Omega - 4\overline{g} = 3(e-3)(e-15) + 9t_2 + 12t_3 + 9t_4. \tag{44}$$

(i) If $e = 16$ or 17 , then $\sigma \leq 12$ and $5\Omega \geq 4\overline{g} + 39 \geq 35$. Thus $\Omega \geq 7$; Then $\frac{(\Omega+5)(\Omega+8)}{9} \geq 20 \geq e \geq \sigma$. In this case, we have the inequality (42).

(ii) If $e = 15$, then $\sigma = 10$ and $5\Omega = 9t_2 + 12t_3 + 9t_4 + 4\overline{g}$.

(aa) But if $\Omega \geq 4$, then $\frac{(\Omega+5)(\Omega+8)}{9} \geq \Omega + 8 \geq 12 \geq 10 = \sigma$. Moreover, $\frac{(\Omega+5)(\Omega+8)}{9} \geq \frac{2}{3} \times 18 > \frac{2}{3}e$. In this case, we have the inequality (42) and

$$(1 + \frac{1}{2})\frac{(\Omega+5)(\Omega+8)}{9} \geq e. \tag{45}$$

(bb) If $\Omega = 3$ or 2 , then $5\Omega = 9t_2 + 12t_3 + 9t_4 + 4\overline{g}$ has no solution.

(2) $\nu_1 = 4$. Hence, $8 \leq \sigma \leq e - \nu_1 \leq e - 4 \leq 13$. Hence, $e = 12$ or 13.

We consider the plane type $[e; 4^{t_4}, 3^{t_3}, 2^{t_2}]$. Then it is easy to get the formula:

$$2\Omega - \overline{g} = \frac{3}{2}(e-3)(e-12) + 2t_2 + 3t_3 \geq 0. \tag{46}$$

We shall examine the following cases separately.

(i) If $17 \geq e \geq 13$, then $\sigma \leq 13$ and $2\Omega \geq -2\overline{g} + 15 \geq 13$. Thus $\Omega \geq 7$ and we get the inequality (42).

(ii) If $e = 12$, then $\sigma \leq 8$. If $\Omega \geq 3$, then $\frac{(\Omega+5)(\Omega+8)}{9} > 9$ and we get the inequality (42).

(iii) If $e \leq 11$, then $3\nu_1 \leq e$ and $\nu_1 \leq 3$, a contradiction.

Note tha also in these cases, one can verify the inequality (45).

(3) $\nu_1 \leq 3$. We consider the plane type $[e; 3^{t_3}, 2^{t_2}]$. It is easy to get the formula: $\Omega = (e-3)(e-9) + t_2$ and $\overline{g} = \dfrac{e(e-3)}{2} - t_2 - 3t_3$.

10 estimate of e in terms of Ω

10.1 case in which $B \geq 3$

Assume $B \geq 3$ and $\sigma \geq 6$.

From Lemma 1 implying that

$$\sigma A_1 + 4\overline{g} \geq (2e - B\sigma - 2\sigma),$$

it follows that

$$\sigma\Omega_1 + 12\overline{g} \geq 3(2e - B\sigma - 2\sigma).$$

Thus,

$$\sigma\Omega + 2(6-\sigma)\overline{g} \geq 3e(1 - \frac{2}{B})(\sigma - 2) + 3u(1 + \frac{2}{B})(\sigma - 2). \tag{47}$$

10.2 case in which $g > 0$

Assuming $g > 0$, we obtain

$$\sigma\Omega \geq 3e(1 - \frac{2}{B})(\sigma - 2).$$

Thus, the following result is proved.

Proposition 5. *If $g > 0$, $B \geq 3$ and $\sigma \geq 6$, then*

$$e \leq \frac{B\sigma\Omega}{3(B-2)(\sigma-2)}. \tag{48}$$

10.3 case in which $g = 0$

We shall prove the next result.

Proposition 6. *If $g = 0$, $B \geq 3$ and $\sigma \geq 6$, then*

$$e \leq \frac{B\sigma\Omega}{3(B-2)(\sigma-2)}. \tag{49}$$

Proof.
Supposing that $e > \frac{B\sigma\Omega}{3(B-2)(\sigma-2)}$, we shall derive a contradiction.
Then by hypothesis,

$$3e(1 - \frac{2}{B})(\sigma - 2) > \sigma\Omega.$$

From the inequality (47), it follows that

$$\begin{aligned}\sigma\Omega - 2(6-\sigma) &\geq 3e(1 - \frac{2}{B})(\sigma - 2) + 3u(1 + \frac{2}{B})(\sigma - 2) \\ &\geq \sigma\Omega + 3u(1 + \frac{2}{B})(\sigma - 2).\end{aligned}$$

Thus

$$2(6-\sigma) \geq 3u(1 + \frac{2}{B})(\sigma - 2).$$

Hence,

$$2(6-\sigma) \geq 3u(1 + \frac{2}{B})(\sigma - 6 + 4) = 3u(1 + \frac{2}{B})(\sigma - 6 + 4) + 12u(1 + \frac{2}{B}).$$

Then

$$2 \geq 3u(1+\frac{2}{B}) + 12u(1+\frac{2}{B})\frac{1}{\sigma-6} \geq 3u.$$

Therefore, $u = 0$.

Thus, the inequality (47) turns out to be

$$\sigma\Omega - 2(6-\sigma) \geq 3e(1-\frac{2}{B})(\sigma-2).$$

Since $e = B\sigma$, it follows that

$$3e(1-\frac{2}{B})(\sigma-2) = 3\sigma(B-2)(\sigma-2).$$

Hence,

$$\sigma\Omega \geq 2(6-\sigma) + 3\sigma(B-2)(\sigma-2).$$

From this,

$$\Omega \geq 2(1-\frac{6}{\sigma}) + 3(B-2)(\sigma-2).$$

By $\sigma \geq 6$, we have $2(1-\frac{6}{\sigma}) < 1$.

Hence,

$$\Omega \geq 3(B-2)(\sigma-2). \tag{50}$$

However, since $e = B\sigma$, it follows that

$$3(B-2)(\sigma-2) = 3e(1-\frac{2}{B})(1-\frac{2}{\sigma}).$$

Hence,

$$\Omega \geq 3e(1-\frac{2}{B})(1-\frac{2}{\sigma}).$$

Therefore,

$$e \leq \frac{B\sigma\Omega}{3(B-2)(\sigma-2)}.$$

In particular, since $B \geq 3$ and $\sigma \geq 6$, it follows that

$$e \leq \frac{B\sigma\Omega}{3(B-2)(\sigma-2)} \leq \frac{3}{2}\Omega.$$

10.4 case in which $B \leq 2$

Under the assumption $\sigma \geq 7$, from

$$e \leq \varepsilon_B({A_1}^2 + 3A_1 + 2\overline{g} + 4)$$

it follows that

$$e \leq \varepsilon_B(\frac{\Omega_1^2}{9} + \Omega_1 + 2\overline{g} + 4). \tag{51}$$

From this we wish to derive the following estimate:

$$e \leq \varepsilon_B \frac{(\Omega + 5)(\Omega + 8)}{9}. \tag{52}$$

To verify this inequality, recalling that the equality (43), we may assume $\Omega_1 + \overline{g} < 0$ and $g > 0$.

10.5 case when $B = 0$ or $B = 2$

Here we treat the case when $B = 0$ or $B = 2$. To simplify the notation we assume futher that $B = 0$.

Then from Lemma 1 implying that

$$\sigma A_1 + 4\overline{g} \geq 2u(\sigma - 2)$$

it follows that

$$\sigma\Omega_1 + 12\overline{g} \geq 6u(\sigma - 2).$$

Thus,

$$\Omega_1 + \overline{g} \geq 6u(1 - \frac{2}{\sigma}) + (1 - \frac{12}{\sigma})\overline{g}. \tag{53}$$

Thus if $\sigma \geq 12$ and $g > 0$, then $\Omega_1 + \overline{g} \geq 0$, which contradicts the hypothesis.

Thus we may assume $\sigma \leq 11$;hence $\nu_1 \leq 5$. Moreover, by the inequality (53), we can suppose that

$$6u(1 - \frac{2}{\sigma}) + (1 - \frac{12}{\sigma})\overline{g} < 0. \tag{54}$$

(1) $\nu_1 = 5$. The type becomes $[\sigma * e; 5^{t_5}, 4^{t_4}, 3^{t_3}, 2^{t_2}]$. Then

- $\Omega = 2(\sigma - 4)(e - 4) - 4 + t_2 - 3t_4 - 8t_5$,
- $\overline{g} = (\sigma - 1)(e - 1) - 1 - t_2 - 3t_3 - 6t_4 - 10t_5$.

From these, we obtain

$$5\Omega - 4\overline{g} = 6((\sigma - 6)(e - 6) - 36) + 120 + 9t_2 + 12t_3 + 9t_4. \qquad (55)$$

Therefore, if $\sigma = 10$ or 11, then $5\Omega - 4\overline{g} \geq 0$.

If $\sigma = 10$ then by the inequality (54) we have

$$6u(1 - \frac{1}{5}) + (1 - \frac{6}{5})\overline{g} < 0.$$

Hence ,

$$24u < \overline{g} \leq \frac{5\Omega}{4} \leq 2\Omega.$$

Thus

$$12u < \Omega.$$

If $\sigma = 11$ then by the inequality (54) we have

$$6u(1 - \frac{2}{11}) + (1 - \frac{12}{11})\overline{g} < 0.$$

Hence ,

$$54u < \overline{g} < \frac{5\Omega}{4}.$$

Thus

$$12u < \Omega.$$

From this we shall verify (52), where $\sigma = 10$ or 11.
If $u = 0$ then $e = \sigma$ and we are done.
If $u > 0$ then $e = \sigma + u$, $\Omega \geq 12u \geq 10$ and

$$\frac{(\Omega + 5)(\Omega + 8)}{9} \geq 2(\Omega + 5) \geq 2(12u + 5) = 10 + u + 23u > 11 + u > e.$$

This implies the required inequality.

10.6 case when $B = 1$

Assume that $B = 1$. Then $2e - 3\sigma = 2u - p$.
case 1). Suppose that $2e - 3\sigma = 2u - p \geq 0$.
From Lemma 1 implying that

$$\sigma A_1 + 4\overline{g} \geq (2u - p)(\sigma - 2)$$

it follows that

$$\sigma\Omega_1 + 12\overline{g} \geq 3(2u-p)(\sigma-2).$$

Thus,

$$\Omega_1 + \overline{g} \geq 3(2u-p)(1-\frac{2}{\sigma}) + (1-\frac{12}{\sigma})\overline{g}. \tag{56}$$

Therefore, if $\sigma \geq 12$ and $g > 0$, then $\Omega_1 + \overline{g} \geq 0$, which contradicts the hypothesis. Hence, $\sigma \leq 11$, which implies that $\nu_1 \leq 5$.

Moreover, by the inequality (56), we can suppose

$$3(2u-p)(1-\frac{2}{\sigma}) + (1-\frac{12}{\sigma})\overline{g} < 0. \tag{57}$$

We distinguish the various cases according to the value of ν_1.

1) $\nu_1 = 5$.

Then since $\sigma = 10$ or 11, we have $p = 1$ or 0. Thus

$$e = \sigma + u + \nu_1 = p + u + 3\nu_1 \leq 1 + u + 3\nu_1.$$

By $2u - p \geq 0$, we have $u > 0$ when $p = 1$. Thus if $u = 0$ then $p = 0$ and so $e = 3\nu_1 = \varepsilon_1(\sigma)$. Therefore, the inequality (52) has been proved.

Writing σ as $10 + \varepsilon$, where $\varepsilon = 0$ or 1 , from (57), it follows that

$$3(2u-p)(1-\frac{2}{10+\varepsilon}) + (1-\frac{12}{10+\varepsilon})\overline{g} < 0.$$

Then

$$\overline{g} > 3(2u-p)\frac{8+\varepsilon}{2-\varepsilon} \geq 12(2u-p).$$

By $5\Omega - 4\overline{g} \geq 0$ we get

$$5\Omega \geq 4\overline{g} \geq 48(2u-p).$$

Hence,

$$\Omega \geq\geq \frac{48(2u-p)}{5} > 8(2u-p).$$

Thus, $\Omega + 5 > 16u - 8 + 5 = 16u - 3 > 9$ and $\Omega + 8 > 16u > 15u$. Therefore,

$$\varepsilon_1 \frac{(\Omega+5)(\Omega+8)}{9} > \varepsilon_1(16u) = 24u > e = u + 3\nu_1 + p.$$

2) $\nu_1 = 4$.

Then since $8 \leq \sigma \leq 11$, we have $p \leq 3$. Thus

$$e = \sigma + u + \nu_1 = p + u + 3\nu_1 \leq p + u + 3\nu_1.$$

By $2u - p \geq 0$, we have $u > \frac{p}{2}$.

Thus if $u = 0$ then $p = 0$ and so $e = 3\nu_1 = \varepsilon_1(\sigma)$.

Writing σ as $10 + \varepsilon$, where $\varepsilon = 0$ or 1 , from (57), it follows that

case 2). Suppose that $2e - 3\sigma = 2u - p < 0$.

In this case, we have already established the estimate.

11 appendix

Table 1: $A = 1, 2$

A	σ	type	genus
0	6	$[6*6;3^8]$	1
1	6	$[6*7;3^{10}]$	0
1	6	$[6*6;3^7,2^4]$	0
1	6	$[6*6;3^7,2^3]$	1
1	6	$[6*6;3^7,2^2]$	2
1	6	$[6*6;3^7,2]$	3
1	6	$[6*6;3^7]$	4
1	8	$[8*8;4^7,3^2]$	1
1	8	$[8*8;4^7,3^2,2]$	0
1	10	$[10*11;5^9]$	0
1	12	$[12*12;6^7,5,4]$	0
2	6	$[6*7;3^9]$	3
2	6	$[6*7;3^9,2]$	2
2	6	$[6*7;3^9,2^2]$	1
2	6	$[6*7;3^9,2^3]$	0
2	6	$[6*6;3^6,2^7]$	0
2	6	$[6*6;3^6,2^6]$	1
2	6	$[6*6;3^6,2^5]$	2
2	6	$[6*6;3^6,2^4]$	3
2	6	$[6*6;3^6,2^3]$	4
2	6	$[6*6;3^6,2^2]$	5
2	6	$[6*6;3^6,2]$	6
2	6	$[6*6;3^6]$	7
2	7	$[7*10,1;3^{11}]$	0

Table 2: $A = 2$(continued)

A	σ	type	genus
2	8	$[8 * 8; 4^6, 3^4]$	1
2	8	$[8 * 8; 4^6, 3^4, 2]$	0
2	8	$[8 * 9; 4^9]$	2
2	8	$[8 * 9; 4^9, 2]$	1
2	8	$[8 * 9; 4^9, 2^2]$	0
2	8	$[8 * 8; 4^7, 3, 2^4]$	0
2	8	$[8 * 8; 4^7, 3, 2^3]$	1
2	8	$[8 * 8; 4^7, 3, 2^2]$	2
2	8	$[8 * 8; 4^7, 3, 2]$	3
2	8	$[8 * 8; 4^7, 3]$	4
2	9	$[9 * 13, 1; 4^{10}]$	0
2	10	$[10 * 10; 5^6, 4^3, 3]$	0
2	10	$[10 * 10; 5^7, 4, 3, 2^2]$	0
2	10	$[10 * 10; 5^7, 4, 3, 2]$	1
2	10	$[10 * 10; 5^7, 4, 3]$	2
2	12	$[12 * 12; 6^6, 5^3]$	1
2	12	$[12 * 12; 6^6, 5^3, 2]$	0
2	12	$[12 * 12; 6^7, 5, 3^2]$	0
2	14	$[14 * 14; 7^7, 6, 4]$	1
2	14	$[14 * 14; 7^7, 6, 4, 2]$	0
2	15	$[15 * 22, 1; 7^9]$	0
2	16	$[16 * 16; 8^6, 7^2, 6]$	0
2	20	$[20 * 20; 10^7, 9, 5]$	0

Table 3: $\Omega = 1, 2, 3, 5, 6$

σ	TYPE	Ω	ω	A
7	$[7 * 9, 1; 1]$	1	1	9
7	$[7 * 9, 1; 2^1]$	2	2	9
7	$[7 * 9, 1; 2^2]$	3	3	9
7	$[7 * 9, 1; 2^3]$	4	4	9
10	$[10 * 11; 5^9]$	4	2	1
12	$[12 * 12; 6^7, 5, 4]$	4	2	1
7	$[7 * 9, 1; 2^4]$	5	5	9
10	$[10 * 10; 5^7, 4, 3]$	5	3	2
10	$[10 * 10; 5^7, 4]$	5	3	3
7	$[7 * 9, 1; 2^5]$	6	6	9
10	$[10 * 10; 5^7, 4, 3, 2]$	6	4	2
10	$[10 * 10; 5^7, 4, 2]$	6	4	3
12	$[12 * 12; 6^6, 5^3]$	6	3	2
14	$[14 * 14; 7^7, 6, 4]$	6	3	2

Table 4: $\Omega = 7, 8$

σ	TYPE	Ω	ω	A
7	$[7 * 9, 1; 2^6]$	7	7	9
10	$[10 * 10; 5^6, 4^3, 3]$	7	4	2
10	$[10 * 10; 5^7, 4, 3, 2^2]$	7	5	2
10	$[10 * 10; 5^7, 4, 2^2]$	7	5	3
10	$[10 * 10; 5^6, 4^3]$	7	4	3
12	$[12 * 12; 6^6, 5^3, 2]$	7	4	2
12	$[12 * 12; 6^7, 5, 3^2]$	7	4	2
12	$[12 * 12; 6^7, 5]$	7	4	4
12	$[12 * 12; 6^7, 5, 3]$	7	4	3
14	$[14 * 14; 7^7, 6, 4, 2]$	7	4	2
15	$[15 * 22, 1; 7^9]$	7	3	2
16	$[16 * 16; 8^6, 7^2, 6]$	7	3	2
20	$[20 * 20; 10^7, 9, 5]$	7	3	2
7	$[7 * 9, 1; 2^7]$	8	8	9
8	$[8 * 10, 1; 1]$	8	6	14
10	$[10 * 10; 5^6, 4^3, 2]$	8	5	3
10	$[10 * 10; 5^7, 3^3]$	8	5	3
10	$[10 * 10; 5^7, 4, 2^3]$	8	6	3
10	$[10 * 10; 5^7, 3]$	8	5	5
10	$[10 * 10; 5^7, 3^2]$	8	5	4
12	$[12 * 13; 6^8, 5]$	8	4	3
12	$[12 * 12; 6^7, 5, 3, 2]$	8	5	3
12	$[12 * 12; 6^7, 5, 2]$	8	5	4
14	$[14 * 14; 7^7, 5^2]$	8	4	3
16	$[16 * 16; 8^7, 7, 4]$	8	4	3

Table 5: $\Omega = 9$

σ	TYPE	Ω	ω	A
7	$[7 * 9, 1; 2^8]$	9	9	9
8	$[8 * 10, 1; 2]$	9	7	14
10	$[10 * 10; 5^5, 4^5]$	9	5	3
10	$[10 * 11; 5^8, 4, 3]$	9	5	3
10	$[10 * 10; 5^6, 4^3, 2^2]$	9	6	3
10	$[10 * 10; 5^7, 3^3, 2]$	9	6	3
10	$[10 * 10; 5^7, 4, 2^4]$	9	7	3
10	$[10 * 10; 5^7, 3, 2]$	9	6	5
10	$[10 * 10; 5^7, 3^2, 2]$	9	6	4
10	$[10 * 11; 5^8, 4]$	9	5	4
12	$[12 * 12; 6^7, 4^2, 3]$	9	5	3
12	$[12 * 13; 6^8, 5, 2]$	9	5	3
12	$[12 * 12; 6^7, 5, 3, 2^2]$	9	6	3
12	$[12 * 12; 6^7, 5, 2^2]$	9	6	4
12	$[12 * 12; 6^7, 4^2]$	9	5	4
14	$[14 * 14; 7^7, 5^2, 2]$	9	5	3
14	$[14 * 14; 7^7, 6, 3^2]$	9	5	3
14	$[14 * 14; 7^7, 6]$	9	5	5
14	$[14 * 14; 7^7, 6, 3]$	9	5	4
16	$[16 * 16; 8^5, 7^4]$	9	4	3
16	$[16 * 17; 8^8, 6]$	9	4	3
16	$[16 * 16; 8^7, 7, 4, 2]$	9	5	3
18	$[18 * 18; 9^7, 7, 6]$	9	4	3
22	$[22 * 22; 11^7, 10, 5]$	9	4	3

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