LECTURE ON THE NONBIRATIONAL INVARIANT E OF ALGEBRAIC PLANE CURVES

SHIGERU IITAKA
GAKUSHUIN UNIVERISTY

1. Introduction

2. History

Let me start with recalling the history of algebraic plane curves from the viewpoint of birational geometry.

2.1. **Q.T.** The fundamental quadratic transformation T between the projective plane $\mathbf{P}^2$ is defined by

$$Y_0 = X_1 X_2, \quad Y_1 = X_0 X_2, \quad Y_2 = X_1 X_0.$$

- $X = \frac{Y_1}{Y_0} = \frac{X_0}{X_1} = \frac{1}{x}$,
- $Y = \frac{Y_2}{Y_0} = \frac{X_0}{X_2} = \frac{1}{y}$.

By Q.T, a plane curve C of degree d is transformed into a plane curve C'.

Ex. If C is a quartic with three double points P_0, P_1, P_2, then C' turns out to be a conic.

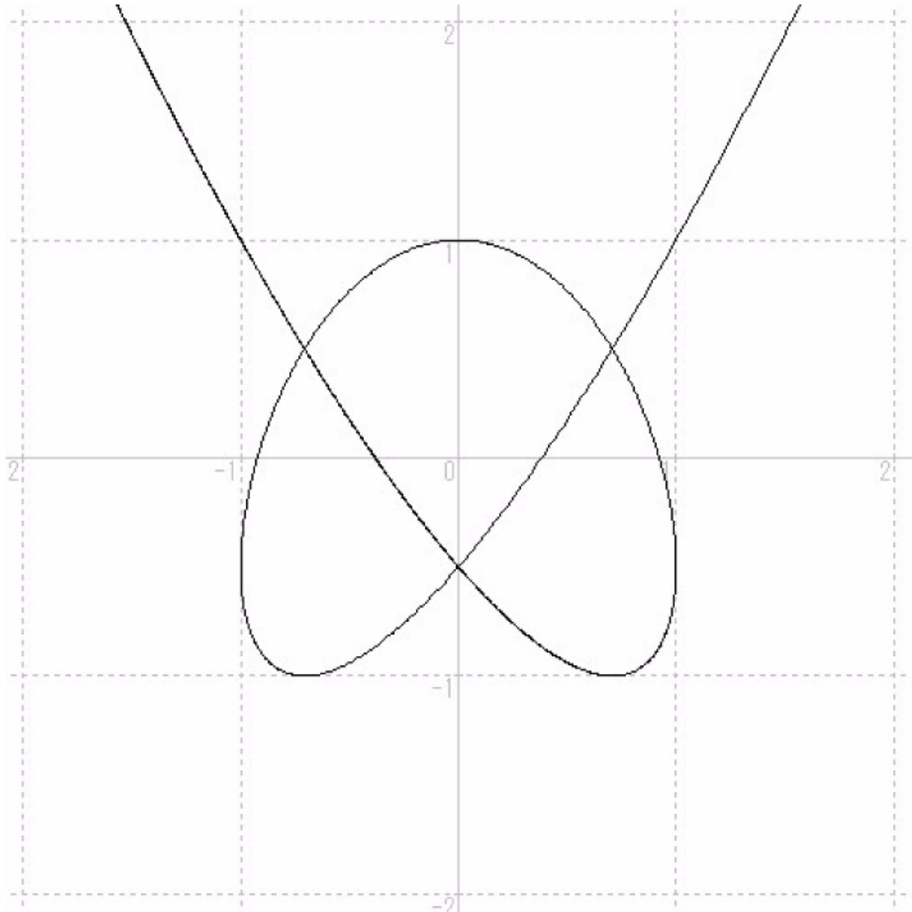


FIGURE 1. degree 4 curve with 3 nodes

2.2. **Noether's formula.** Suppose that C is a curve of degree d with singular points of which multiplicites are $\nu_0, \nu_1, \nu_2, \cdots, \nu_r$ such that $\nu_0 \geq \nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$.

If C has singular points with multiplicities ν_0, ν_1, ν_2 at P_0, P_1, P_2,then we see that C' has multiplicities ${\nu_0}', \nu_1', {\nu_2}'$ at Q_0, Q_1, Q_2 where

$$\nu_0' = d - \nu_1 - \nu_2, \quad \nu_1' = d - \nu_0 - \nu_2, \quad \nu_2' = d - \nu_1 - \nu_2$$

.

Here, $d' = 2d - \nu_0 - \nu_1 - \nu_2$. (Noether's formula)

In particular, if $d' < d$,then C' looks much simpler than C.

$d' < d$ if and only $d < \nu_0 + \nu_1 + \nu_2$.

$d < \nu_0 + \nu_1 + \nu_2$ is an inequality called Noether's inequality.

Example. A curve defined by $x = \cos 5\theta, y = \cos 4\theta$ is a rational curve of degree 5 with 6 double points.

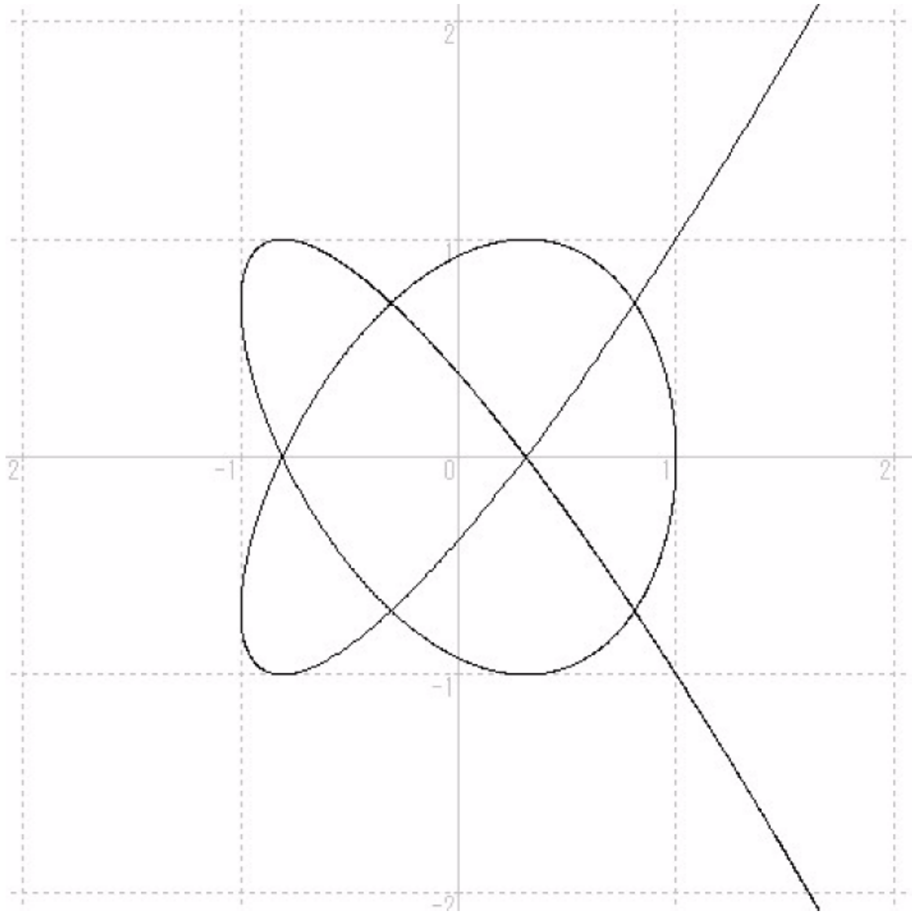
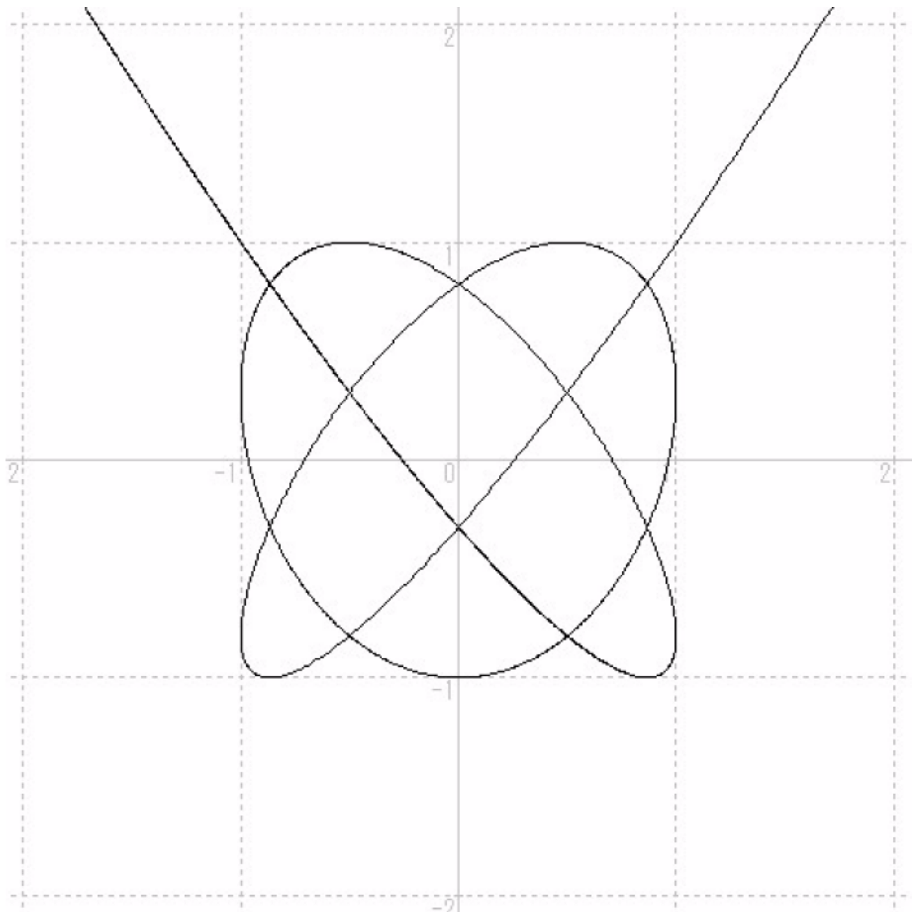


FIGURE 2

Then $\nu_0 = \nu_1 = \nu_2 = 2$ and $d' = 2d - \nu_0 - \nu_1 - \nu_2 = 10 - 6 = 4$.

This is transformed into a line by a birational transformation between the projective plane $\mathbf{P}^2$, which we call Cremona transformation.

However, the curve defined by $x = \cos 6\theta, y = \cos 5\theta$ has degree 6 and 10 double points.



Since $\nu_0 = \nu_1 = \nu_2 = 2$, it follows that $d' = 2d - \nu_0 - \nu_1 - \nu_2 = 12 - 6 = 6$.

Thus C' does not look simpler than C.

Fact 1. If C is a curve tranformed from a line on $\mathbf{P}^2$ by a Cremona transformation then $d < \nu_0 + \nu_1 + \nu_2$.

But the converse is not true.

Fact 2. If $d \geq \nu_0 + \nu_1 + \nu_2$, C cannot be tranformed into a line on $\mathbf{P}^2$ by any Cremona transformation.

Fact 3. If C is a curve tranformed from a line on $\mathbf{P}^2$ by a Cremona transformation then $d < \nu_0 + 2\nu_2$.

These are proven by using logarithmic Kodaira dimension.

$d \geq \nu_0 + \nu_1 + \nu_2$ is the converse of Noether's inequality.

Theorem 1 (Cremona)**.** *If $d \geq \nu_0 + \nu_1 + \nu_2$ holds, then C is Cremonian minimal; If C' is obtained from C by a Cremona transformation, then the degree of $C' \geq d$.*

Just by simple computation.

But even if C is not transformed into line on $\mathbf{P}^2$, after a Cremona tranformation the condition $d \geq \nu_0+\nu_1+\nu_2$ is not necessary satisfied.

For a singular rational curve C on $\mathbf{P}^2$, after a finite number of blowing ups we get a nonsingular rational surface S and a nonsingular curve D which is the proper transform of C.

Say C has degree d and singular points with multiplicites $\nu_0, \nu_1, \nu_2, \cdots, \nu_r$ such that $\nu_0 \geq \nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$.

Then

$$D^2 = d^2 - \nu_0^2 - \nu_1^2 - \cdots - \nu_r^2.$$

In this case , we say that C has the plane type $[d; \nu_0, \nu_1, \nu_2, \cdots, \nu_r]$.

2.3. **Nagata.** In 1960, M.Nagata showed that:

Let D be a nonsingular elliptic curve on a nonsingular rational surface S .

Then $D^2 \leq 9$.

If $D^2 = 9$ then (S, D) is transformed into $(\mathbf{P^2}, C_3)$.

Here C_d means a nonsingular plane curve of degree d.

2.4. **Hartshorne.** In 1970,R.Hartshorne showed that if $g > 1$ then $D^2 \leq 4g + 4$.

Moreover, if hyperelliptic curves defined by $y^2 = \Pi_{j=1}^{2g+1}(x - a_j)$ (distinct roots) satisfy $D^2 = 4g + 4$.

Later it was shown that curves with $D^2 = 4g + 4$ turn out to be such hyperelliptic curves or a curve of plane type [4;1].

Indeed, $D^2 = 16, g = 3$ by applying the theory of minimal models.

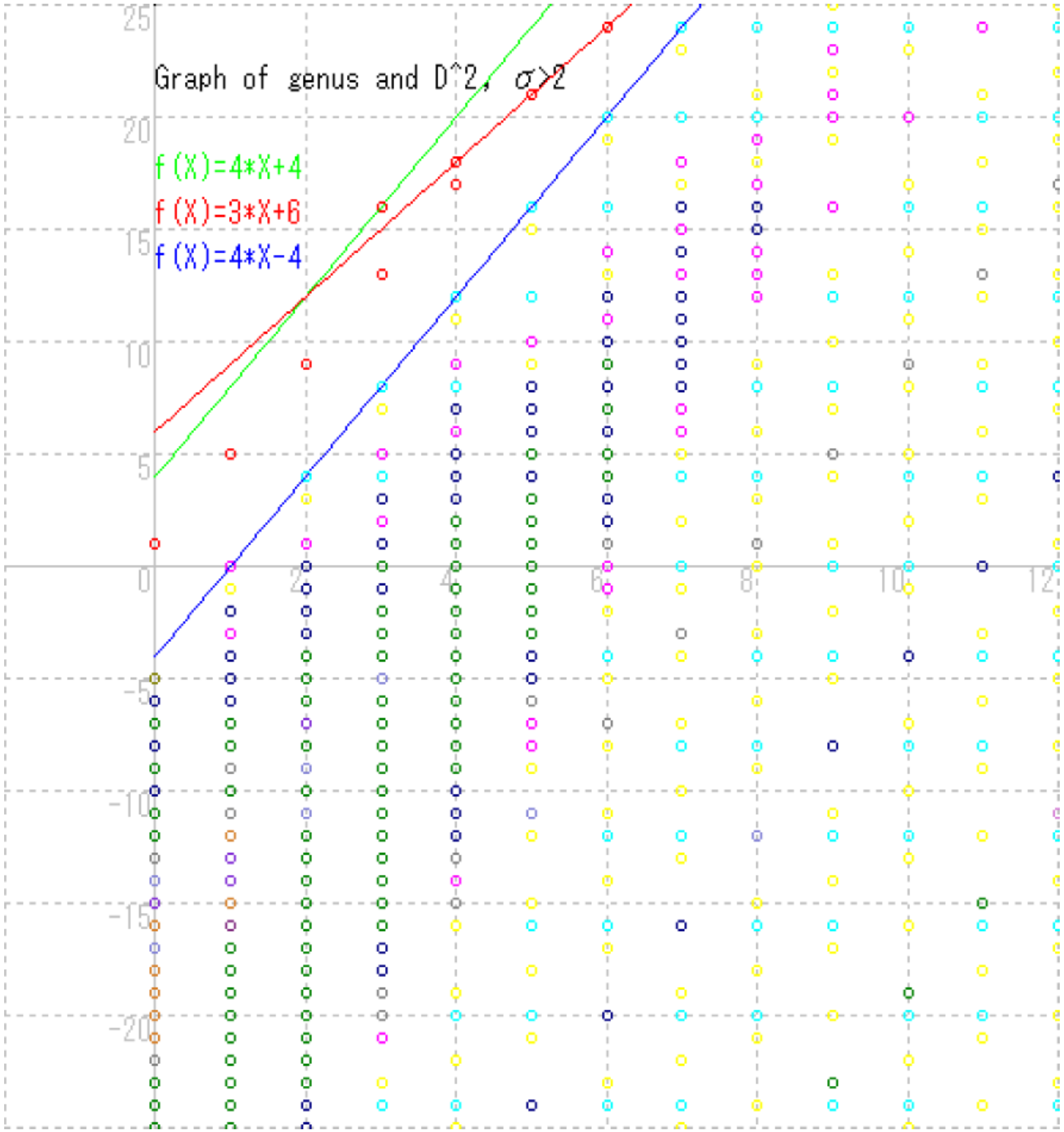

Graph of genus and D^2, σ>2
f(X)=4*X+4
f(X)=3*X+6
f(X)=4*X-4
25
20
15
10
5
0
-5
-10
-15
-20
2
4
6
8
10
12

Figure 4

2.5. **Coolidge.** In 1928,Coolidge studied plane curves C and introduced the notion called the adjoint systems of special index j, which are defined to be $jK_S + D$, $j > 1$.

Coolidge:
If D is rational and $|2K_S + D| = \emptyset$, then D is tranformed into a line on $\mathbf{P}^2$ by a Cremona transformation

This looks like Castelnuovo's criterion of rationality, which claims S is a rational surface if and only if $P_2(S) = 0, \dim H^0(S, \Omega^1) = 0$.

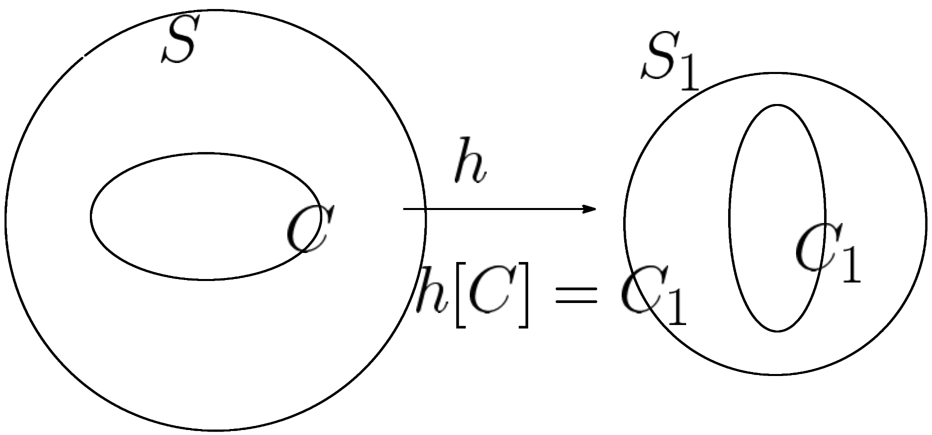
American geometers developed Cremonian geometry.

They studied properties of plane curves which are invariant under Cremona transformations.

In Cremonian geometry, pairs (S, C) of nonsingular rational surfaces S and curves $C \subset S$ are objects of the study.

For pairs (S, C) where C is nonsingular, D stands for C.

Given pairs (S, C) and (S_1, C_1), if there exists a birational map $h : S \to S_1$ such that the proper transform $h[C] = C_1$ then they are said to be *birationally equivalent*.



Let S be a nonsingular rational surface S and D a nonsingular curve on S.

Suppose that $m \geq a \geq 1$.

Then $P_{m,a}[D] = \dim |mK_S + aD| + 1$ are called mixed plurigenera, which depend on S and D.

Z stands for $K_S + D$.

$P_{m,m}[D] = \dim |mZ| + 1$ are logarithmic plurigenera of $S - D$, from which logarithmic Kodaira dimension is introduced, denoted by $\kappa[D]$.

2.6. **minimal models.** A non-singular pair (S, D) is said to be relatively minimal, whenever $D \cdot E \geq 2$ for any (-1) curve E, i.e. ,i.e. an exceptional curve of the first kind, on S such that $E \neq D$.

Theorem 2. *If (S, D) is relatively minimal and $\kappa[D] = 2$ then (S, D) is actually minimal.*

In other words, any birational map from (S_1, D_1) to (S, D) turns out to be regular.

Proposition 1. *Suppose that (S, D) is minimal.*

(1) *If $g = g(D) > 0$ then $Z = K_S + D$ is nef. Moreover, when $\kappa[D] = 2$, Z is big.*

(2) *If $g = 0$ and $\kappa[D] = 2$ then (1) $D^2 \leq -5$ and (2) $Z_\beta = Z - \frac{2}{\beta}D$ is nef and big , where $\beta = -D^2$.*

If $S \neq \mathbf{P^2}$, then there exists a surjective morphism $pr : S \to \mathbf{P^1}$ whose general fibers are $\mathbf{P^1}$. The minimum of the mapping degree of $pr|_D : D \to \mathbf{P^1}$ is denoted by σ.

$P_{1,1}[D] = g$ the genus of D, and $\overline{g}$ is defined to be $g - 1$.
If $\sigma > 4$ then $D + 2K_S$ is nef and big;
$P_{2,1}[D] = Z^2 - \overline{g} + 1 = A + 1$, where $\overline{g} = g - 1$, $A = Z^2 - \overline{g}$;

If $\sigma > 6$ then $|D + 3K_S| \neq \emptyset$ and

$$P_{3,1}[D] = 3Z^2 + 1 - 7\overline{g} + D^2 = 3A - \alpha + 1 = \Omega - \omega + 1$$

where $\alpha = 4\overline{g} - D^2, \Omega = (3Z - 2D) \cdot Z = 3Z^2 - 4\overline{g}$ and $\omega = 3\overline{g} - D^2$.

Problem : Study birational invariants such as D^2, Z^2, α, A, Ω, ω, σ.

TABLE 1. Yii and Yang

Yii	(陰)	$D^2, \alpha = 4\overline{g} - D^2, \omega = 3\overline{g} - D^2, \omega_1 = \omega - \overline{g}$
Yang	(陽)	$Z^2, A = Z^2 - \overline{g}, \Omega = 3Z^2 - 4\overline{g}, A_1 = A - \overline{g}$
Neutral	(中立)	$\sigma, Q = (2Z - D)^2, K_S^2$

Table 2. like elementary particles

1st generation	$d, \nu_1, \nu_2, \cdots, \nu_r, \sigma, e, B$
2nd generation	g =genus,
3rd generation	$\alpha, \omega, \alpha_1, \omega_1, A, \omega, A_1, \Omega_1$
	$P_{2,1}[D], P_{3,1}[D], P_{m,a}[D],$

$\alpha, \omega, A, \Omega, P_{2,1}[D], P_{3,1}[D]$ are very powerful invariants;
They determine the structure of D on S.

If $B \geq 3$, then

- $\sigma \leq 3\omega$
- $\sigma \leq 3\alpha + 3$

If $B \leq 2$, then

- $\sigma \leq (\omega + 1)(\omega + 2)$
- $\sigma \leq (\alpha + 2)(\alpha + 3)$ (By Matsuda)
- $\sigma \leq \omega_1^2 + \omega_1 + 2\overline{g} + 2$
- $\sigma \leq (A + 2)(A + 3)$
- $\sigma \leq {A_1}^2 + 2\overline{g} + 3A_1 + 4$
- $9\sigma \leq {\Omega_1}^2 + 9\Omega_1 + 18\overline{g} + 36.$
- $\sigma \leq \frac{(\Omega+5)(\Omega+8)}{9}$

Define ε_B to be 1(if $B = 0$), $= \frac{3}{2}$(if $B = 1$),=2(if $B = 2$).

- $e \leq \varepsilon_B(\omega + 1)(\omega + 2)$
- $e \leq \varepsilon_B(\alpha + 2)(\alpha + 3)$
- $e \leq \varepsilon_B(\omega_1^2 + \omega_1 + 2\overline{g} + 2)$
- $e \leq \varepsilon_B(A + 2)(A + 3)$

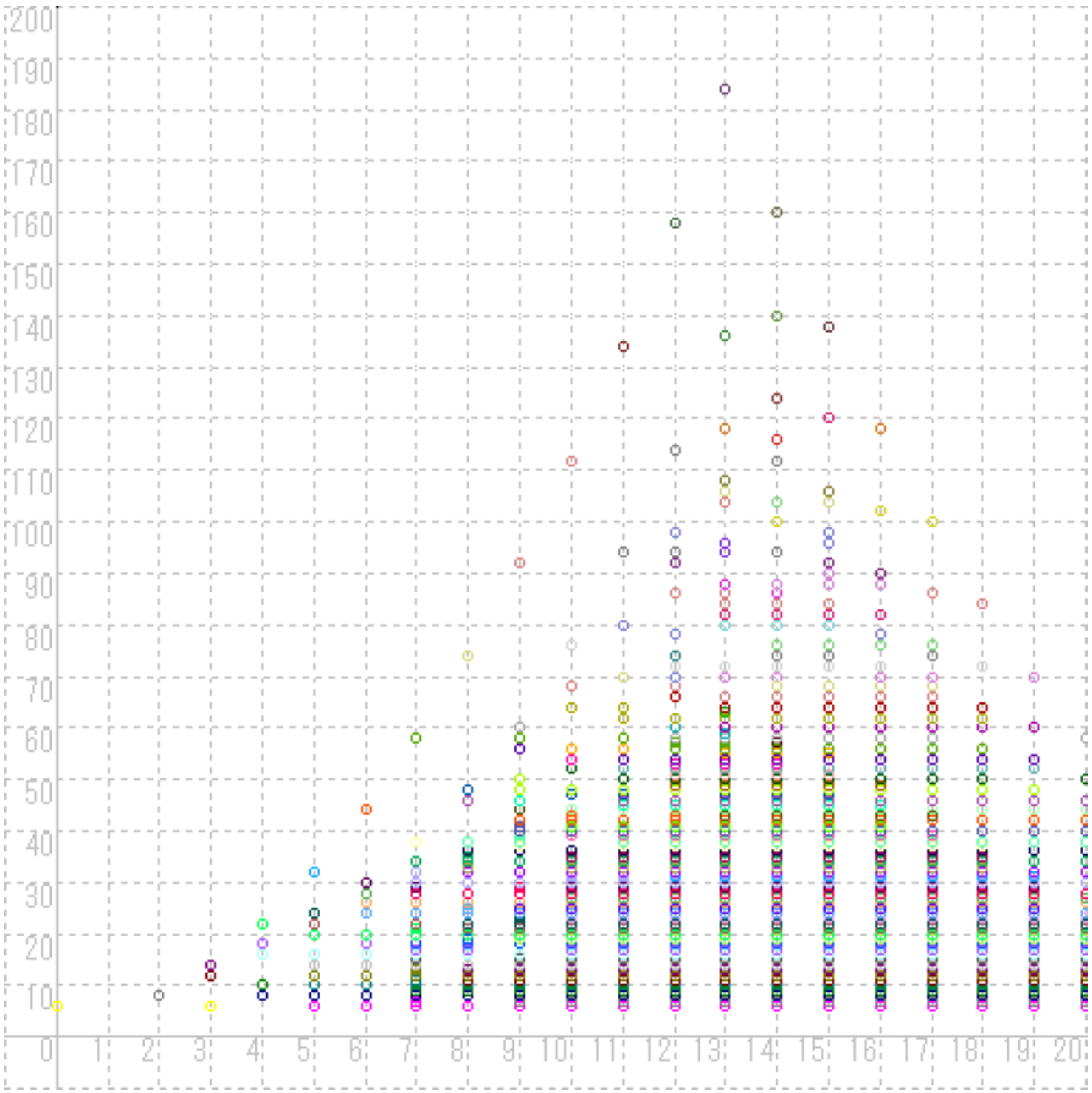


FIGURE 5. iitaka: (α, σ)

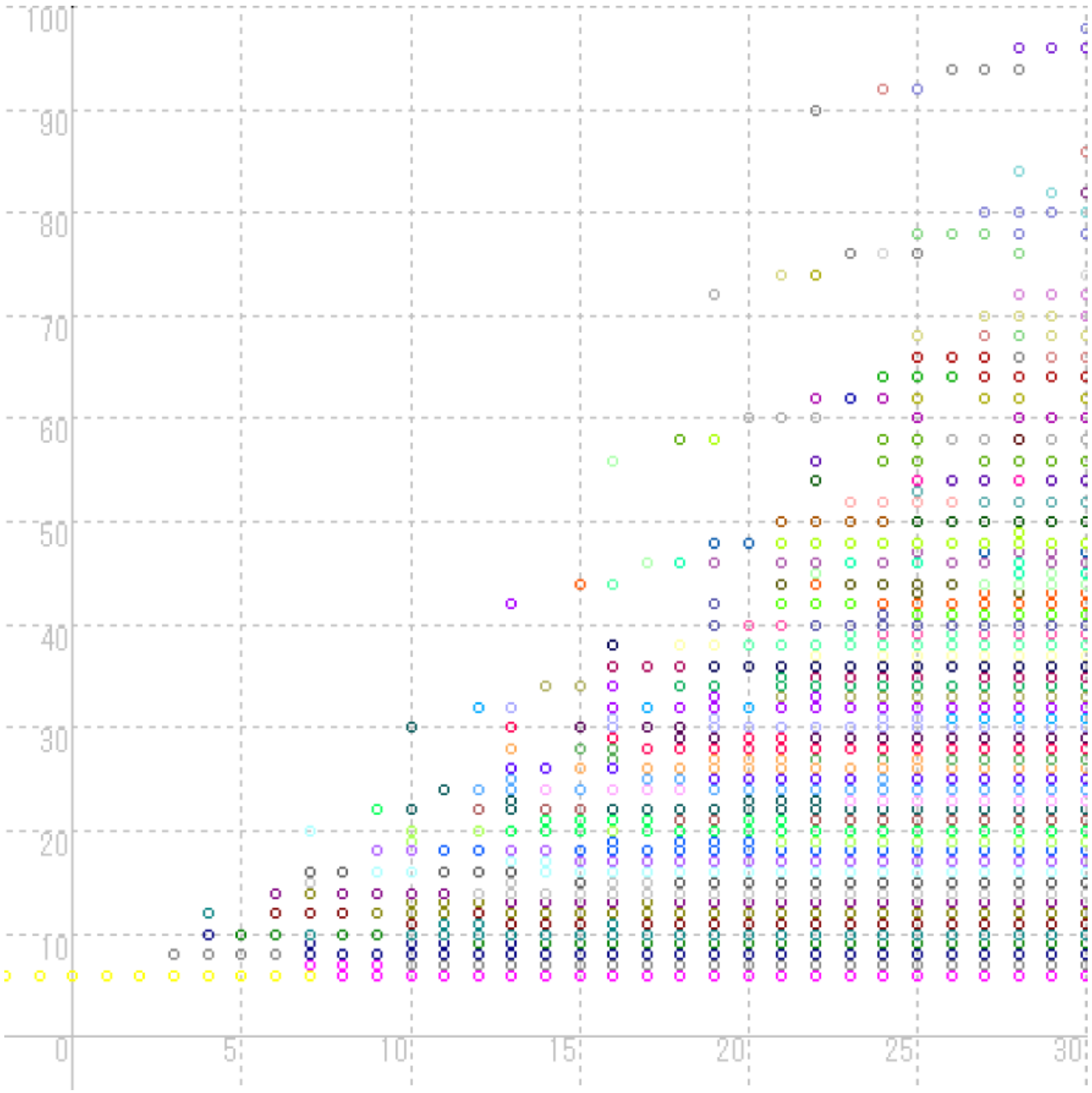


FIGURE 6. iitaka: (ω, σ)

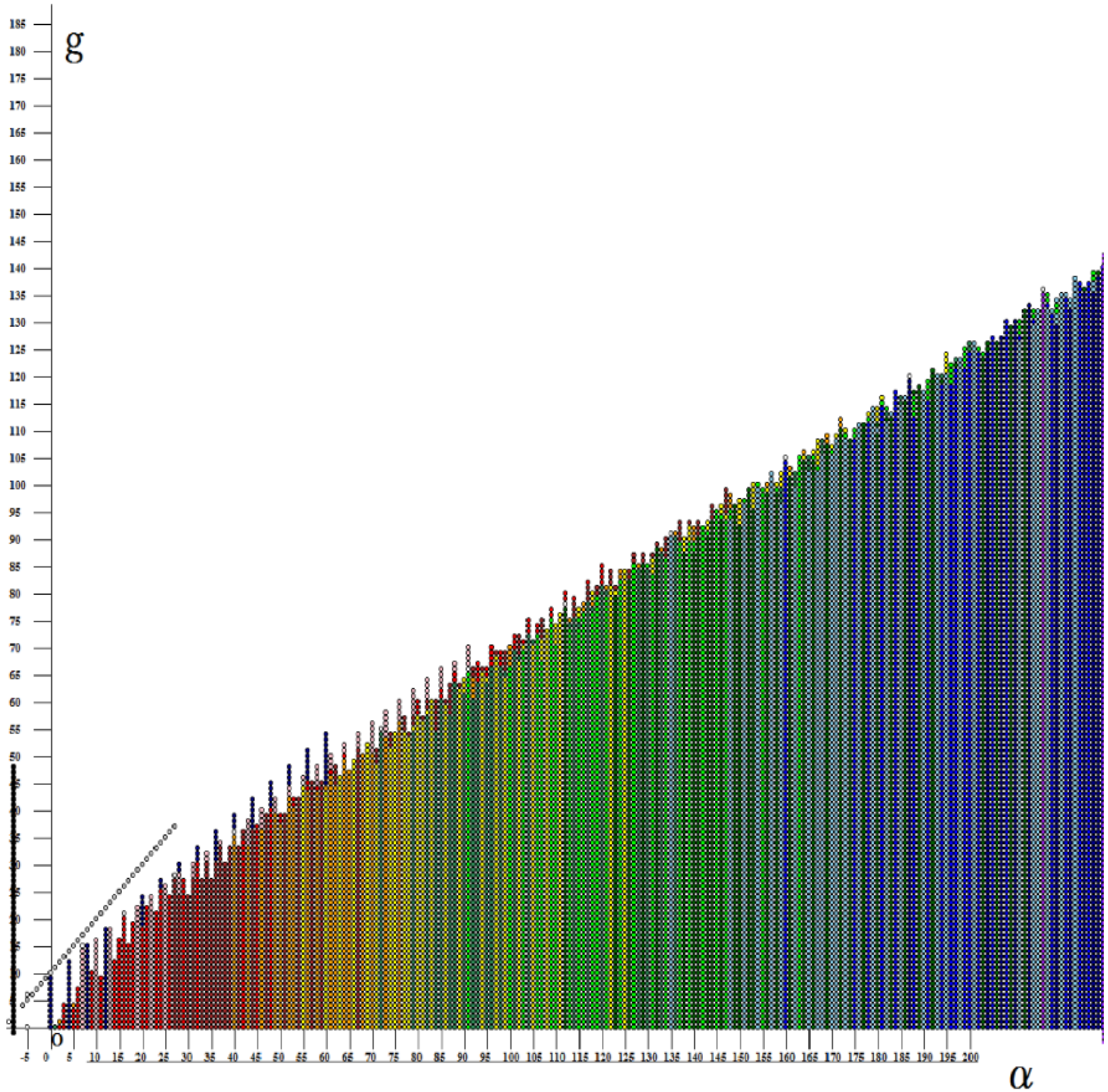


FIGURE 7. Moriyama: (α, g)

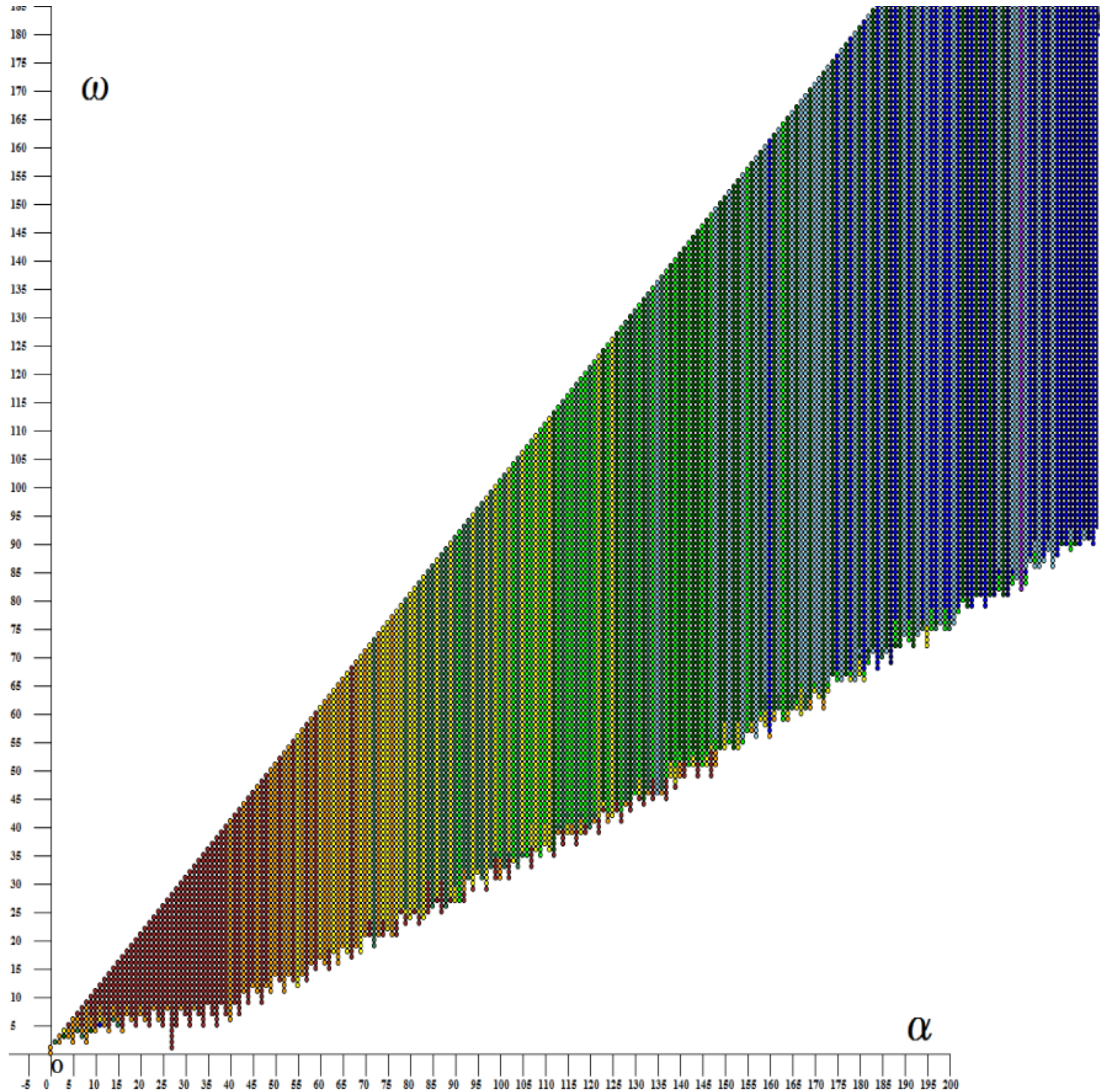


FIGURE 8. Moriyama: (α, ω)

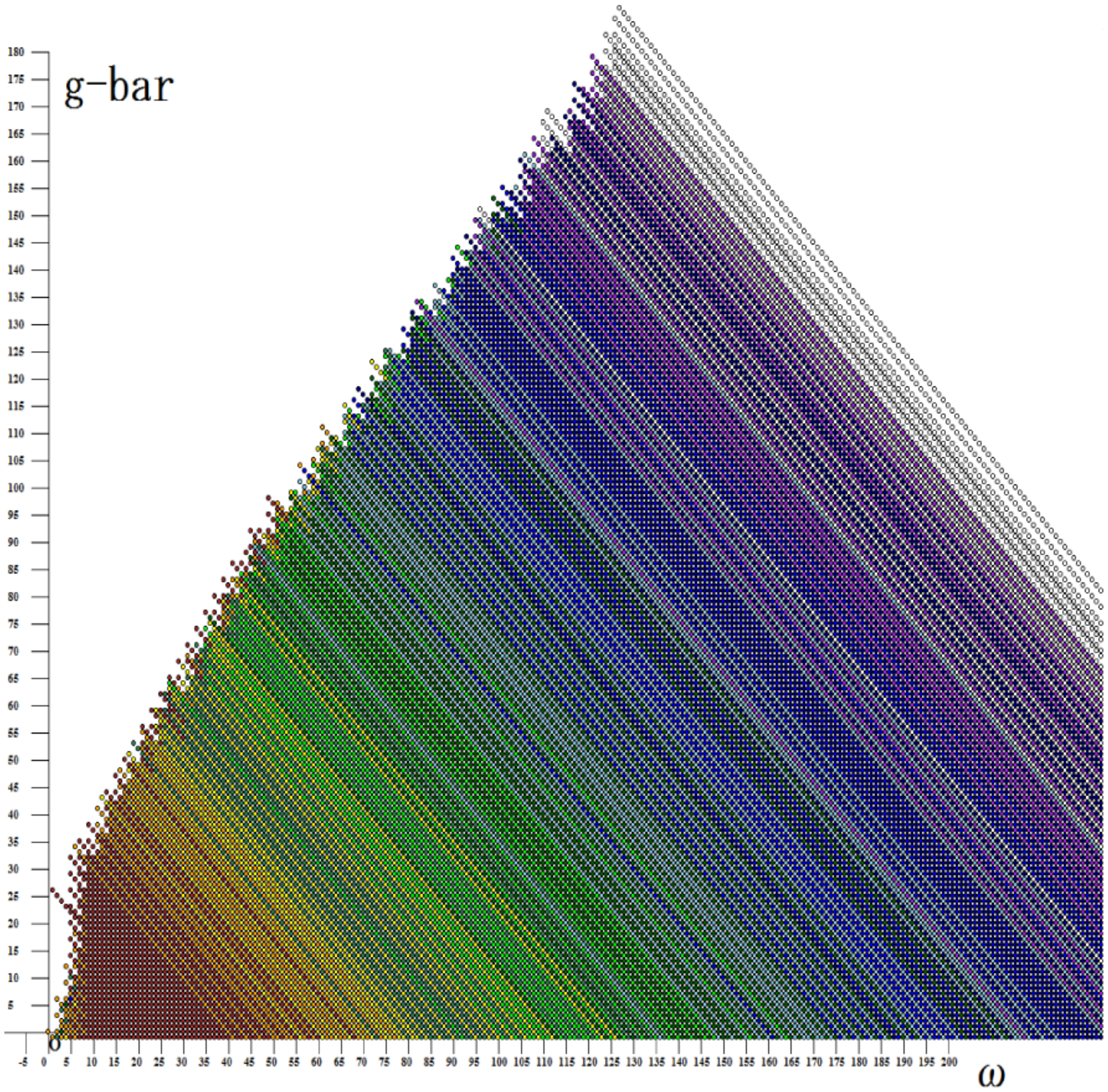
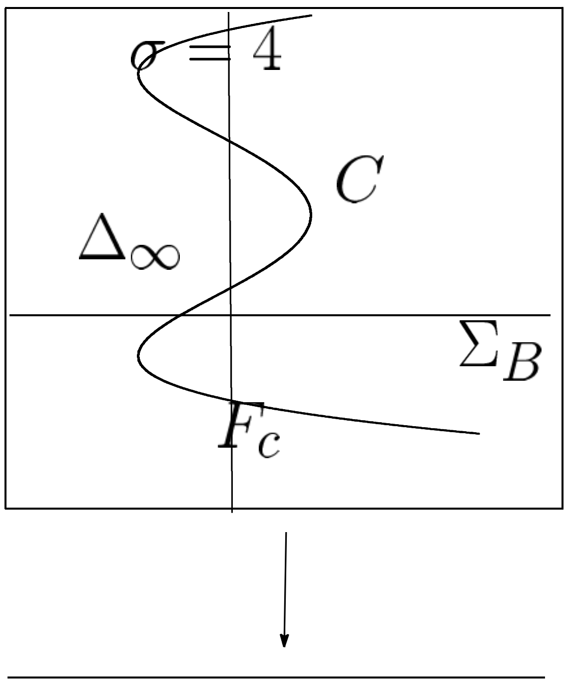


FIGURE 9. Moriyama: (ω, g)

2.7. # **minimal pairs.** Minimal pairs are obtained from some kind of singular models,namely, # minimal pairs.

Any $\mathbf{P}^1-$ bundle over $\mathbf{P}^1$ has a section Δ_∞ with negative self intersection number, which is denoted by a symbol Σ_B, where $-B = {\Delta_\infty}^2$ if $B > 0$.

Σ_B is said to be a Hirzebruch surface of degree B after Kodaira.



Let C be a curve on Σ_B.
Then $C \sim \sigma\Delta_\infty + eF_c$, for some σ and e .
Here, the symbol $\sim$ means the linear equivalence.

$C \cdot F_c = \sigma$ and $C \cdot \Delta_\infty = e - B \cdot \sigma$.
Hereafter, suppose that $C \neq \Delta_\infty$.
Thus $C \cdot \Delta_\infty \geq 0$ and hence, $e \geq B\sigma$.
If $B > 0$ then ${\Delta_\infty}^2 = -B < 0$ and
such a section Δ_∞ is uniquely determined.

2.8. **types of pairs.** The symbol $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$ is said to be the **type** of (Σ_B, C).

Assume that $\sigma \geq 2\nu_1$ and $e - \sigma \geq B\nu_1$.
Moreover, if $B = \nu_1 = 1$ then assume $e - \sigma > 1$.

When the above conditions are satisfied, the pair (Σ_B, C) is said to be # minimal .

- $p = \sigma - 2\nu_1 \geq 0$,
- If $B = 0$, then $u = e - \sigma \geq 0$.
- If $B = 1$, then $u = e - \sigma - \nu_1 \geq 0$.
- If $B \geq 2$, then $u = e - B\sigma \geq 0$.

Using elementary transformations, we get

Theorem A. *If a pair (S, D) is not transformed into a line by Cremona transformations, then*

(1) (S, D) *is obtained from a # minimal pair(model)* (Σ_B, C) *by shortest resolution of singularities of* C *using blowing ups or;*

(2) $(S, D) = (\mathbf{P^2}, C_d)$, C_d *being a nonsingular curve.*

2.9. **possible type and geometric type.** If the pair (S, D) satisfy $\sigma \geq 5$ and and $Z^2 > 0$ which is derived from a # minimal pair (Σ_B, C) with type $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$, then

$$g = (\sigma - 1)(\widetilde{B} - 2)/2 - \sum_{j=1}^{r} \nu_j(\nu_j - 1)/2 \geq 0,$$

$$Z^2 = (\sigma - 2)(\widetilde{B} - 4) - \sum_{j=1}^{r} (\nu_j - 1)^2 > 0.$$

$$D^2 = \sigma \widetilde{B} - \sum_{j=1}^{r} (\nu_j)^2 > 0.$$

Here, $\widetilde{B} = 2e - B\sigma$.

Theorem B. *Suppose that Suppose that* (S, D) *is obtained from a # minimal pair(model)* (Σ_B, C) *by shortest resolution of singularities of* C.

Then (S, D) *is relatively minimal.*

In other words,for any (-1) *curve* Γ *on* S, $\Gamma \cdot \Delta \geq 2$.

Theorem 3. *Suppose that* (S, D) *is relatively minimal and* $\kappa[D] \geq 0$. *If* $g > 0$ *then* $Z = D + K_S$ *is nef.*

2.10. **graph.**

We have the following inequality, which is closely related to the inequality (1).

Theorem 4. *Let $\overline{g}$ denote $g-1$.*

If $\sigma \geq 7$, then

$$\sigma \leq \omega_1^2 + \omega_1 + 2 + 2\overline{g}. \tag{1}$$

Here, $\omega_1 = \omega - \overline{g}$ and $\omega_1 = K_S \cdot D$.

2.11. **lemma.**

Lemma 1. *Suppose that* (S, D) *is minimal.*
(1) *If* $B = 0$ *or 2 then* $2\sigma\overline{g} - (\sigma - 2)D^2 = (2D + \sigma K_S) \cdot D \geq 0$.
(2) *If* $B = 1$ *then either* $(2D+\sigma K_S)\cdot D \geq 0$ *or* $(3D+eK_S)\cdot D \geq 2$.
(3) *If* $B \geq 2$ *then* $(2D + \sigma K_S) \cdot D \geq \sigma^2(B - 2)$.

3. ESTIMATE OF e

3.1. case in which $B \geq 3$.

Lemma 2.

(1) *If* $B \geq 2$ *then* $(2D+\sigma K_S) \cdot D \geq (2e - \sigma B - 2\sigma)\sigma \geq \sigma^2(B-2)$.
(2) *If* $B \geq 3$ *then* $(2D+\sigma K_S) \cdot D \geq \sigma^2$; *in particular,* $2\sigma\overline{g} - (\sigma-2)D^2 \geq \sigma^2$. *Hence,* $\sigma\omega_1 + 2D^2 \geq \sigma^2$.

By Lemma 2(1), we get

$$(\sigma - 2)\omega_1 + 4\overline{g} = (2D + \sigma K_S) \cdot D \geq (2e - \sigma B - 2\sigma)\sigma,$$

where $\omega_1 = 2\overline{g} - D^2$.

If $B \geq 2$ then $e = \sigma B + u$; thus $\sigma = \frac{e-u}{B}$ and

$$2e - \sigma B - 2\sigma = (1 - \frac{2}{B})e + (1 + \frac{2}{B})u.$$

Therefore,

$$(2) \qquad (\sigma - 2)\omega_1 + 4\overline{g} \geq ((1 - \frac{2}{B})e + (1 + \frac{2}{B})u)\sigma \geq (1 - \frac{2}{B})e\sigma.$$

Hence,

$$(1 - \frac{2}{B})e \leq \frac{\sigma - 2}{\sigma}\omega_1 + \frac{4\overline{g}}{\sigma}.$$

Thus,

$$(1 - \frac{2}{B})e \leq \frac{\sigma - 2}{\sigma}\omega - (1 - \frac{6}{\sigma})\overline{g}.$$

Hence, if $B \geq 3, g > 0$ and $\sigma \geq 6$, then

$$(1 - \frac{2}{B})e \leq \frac{\sigma - 2}{\sigma}\omega,$$

and so

$$e \leq \frac{B}{B-2}\frac{\sigma-2}{\sigma}\omega < \frac{B}{B-2}\omega \leq 3\omega. \tag{3}$$

Thus we obtain the next result:

Proposition 2. *If* $g > 0, \sigma \geq 7$ *and* $B \geq 3$ *then* $(1 - \frac{2}{B})e < \omega$.

3.2. **case when $g = 0$.**

When $g = 0$, we get

$$(\sigma - 2)(\omega + 1) - 4 = (\sigma - 2)\omega_1 + 4\overline{g} \geq ((1 - \frac{2}{B})e + (1 + \frac{2}{B})u)\sigma.$$

From this, it follows that

$$\frac{(\sigma - 2)(\omega + 1) - 4}{\sigma} = (1 - \frac{2}{\sigma})\omega + 1 - \frac{6}{\sigma} \geq (1 - \frac{2}{B})e + (1 + \frac{2}{B})u.$$

We shall prove the following result.

Proposition 3. *If $g = 0, \sigma \geq 7$ and $B \geq 3$ then $(1 - \frac{2}{B})e < \omega$.*

Proof.

Supposing that $(1 - \frac{2}{B})e \geq \omega$, we shall derive $u = 0$. Actually, by hypothesis,

$$\frac{(\sigma - 2)(\omega + 1) - 4}{\sigma} = (1 - \frac{2}{\sigma})\omega + 1 - \frac{6}{\sigma} \geq \omega + (1 + \frac{2}{B})u.$$

Thus

$$1 > \frac{2\omega}{\sigma} + \frac{6}{\sigma} + u + \frac{2}{B} > u \geq 0. \tag{4}$$

Therefore, $u = 0$; thus $e = \sigma B$ and so $\sigma = \frac{e}{B}$. Moreover, recalling the inequality (2),

$$(\frac{e}{B} - 2)\omega_1 - 4 = (\sigma - 2)\omega_1 + 4\overline{g} \geq (1 - \frac{2}{B})\frac{e^2}{B},$$

and so

$$(e - 2B)\omega > (1 - \frac{2}{B})e^2 + 6B - e.$$

Finally, we obtain

$$(5) \qquad \omega > (1 - \frac{2}{B})e + 2B - 5 + \frac{4B(B-1)}{e - 2B} > (1 - \frac{2}{B})e.$$

Therefore, if $B \geq 3$,then

$$e < \frac{B}{B-2}\omega \leq 3\omega.$$

q.e.d.

4. FUNDAMENTAL EQUALITIES

By putting $X = \sum_{j=1}^{r} \nu_j^2$ and $Y = \sum_{j=1}^{r} \nu_j$, we obtain

- $X = \widetilde{B}\sigma + \omega - 3\overline{g}$,
- $Y = \widetilde{B} + 2\sigma + \omega - \overline{g}$.

Here, $\widetilde{B} = 2e - \sigma B$.

(1) $B = 0$. Then $\sigma = 2\nu_1 + p, e = \sigma + u$ for some $u \geq 0$ and

- $\widetilde{B} + 2\sigma = 8\nu_1 + 4p + 2u$,
- $\widetilde{B}\sigma = 8\nu_1^2 + 2\nu_1(4p + 2u) + 2pu + 2p^2$.

(2) $B=1$. Then $\sigma=2\nu_1+p, e=\sigma+\nu_1+u$ for $u\geq 0$,

- $\widetilde{B}+2\sigma=8\nu_1+3p+2u$,
- $\widetilde{B}\sigma=8\nu_1^2+2\nu_1(3p+2u)+2pu+p^2$.

(3) $B=2$. Then $\sigma=2\nu_1+p, e=2\sigma+u$ for some $u\geq 0$ and

- $\widetilde{B}+2\sigma=8\nu_1+4p+2u$,
- $\widetilde{B}\sigma=8\nu_1^2+2\nu_1(4p+2u)+2pu+2p^2$.

Defining $w=4-\delta_{1B}$,we get $w=4$ if $B\neq 1$. Further, $w=3$ if $B=1$. Introducing an invariant k by $k=wp+2u$, we have

- $\widetilde{B}+2\sigma=8\nu_1+k$,
- $\widetilde{B}\sigma=8\nu_1^2+2k\nu_1+p(k-2p)$.

Proposition 4. *Letting k be $wp+2u$, w being $4-\delta_{1B}$, we have the following ($B \le 2$) fundamental equalities:*

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $Y = 8\nu_1 + k + \omega_1$.

Here $\tilde{k} = kp - 2p^2$.

4.1. **invariant $\widetilde{\mathcal{Z}}$.** Compute $\widetilde{\mathcal{Z}} = \nu_1 Y - X$.

$$0 \le \widetilde{\mathcal{Z}} = \nu_1(\omega - \overline{g} - k) - \tilde{k} - \omega_1 + 2\overline{g}.$$

$$\widetilde{\mathcal{Z}} = \sum_{j=2}^{\nu_1 - 1} j(\nu_1 - j)t_j \tag{6}$$

Here, t_j denotes the number of $j-$ple singular points. ($\widetilde{\mathcal{Z}}$ equation)

Defining the invariant λ to be $k - \omega_1$, we obtain

$$\nu_1 \lambda \leq -\tilde{k} - \omega_1 + 2\overline{g}. \tag{7}$$

5. INVARIANT A_1

Introduce an invariant A_1 by putting $A_1 = A - \overline{g}$, which satisfies

$$A_1 = \frac{(2Z - D)\cdot Z}{2} - \frac{D\cdot Z}{2} = Z\cdot K_S.$$

By the way,

$$\omega_1 = \omega - \overline{g} = D\cdot K_S;$$

hence, $A_1 - \omega_1 = K_S^2 = 8 - r$.

If $B \leq 2$ then we have the fundamental equalities

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $Y = 8\nu_1 + k + \omega_1$,

from which, we get

$$Y = 8\nu_1 + \omega_1 = 8\nu_1 + A_1 + r - 8 = 8\overline{\nu_1} + r + A_1,$$

where $\overline{\nu_1} = \nu_1 - 1$.

Introduce invariants $\overline{\nu_j}$ and $\overline{Y}$ by $\overline{\nu_j} = \nu_j - 1$ and $\overline{Y} = \sum_{j=1}^{r} \overline{\nu_j}$. Then $\overline{Y} = Y - r$ and

$$\overline{Y} = 8\overline{\nu_1} + k + A_1.$$

Introduce an invariant $\overline{X}$ by

$$\overline{X} = \sum_{j=1}^{r} \overline{\nu_j}^2 = X - 2Y + r,$$

which satisfies that

$$\overline{X} = 8\overline{\nu_1}^2 + 2k\overline{\nu_1} + \tilde{k} - A_1 - 2\overline{g}.$$

Then

$$\mathcal{Z}^* = \overline{\nu_1}\overline{Y} - \overline{X} = \sum_{j=2}^{\nu_1 - 1} (\nu_1 - j)(j-1)t_j.$$

Moreover,denoting $B-2$ by B_2,(here $B > 1$)

$$0 \leq \mathcal{Z}^* = B_2\sigma(2 + \overline{\nu_1} - \sigma) - \overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g}.$$

we obtain

$$B_2\sigma(\sigma - 2 - \overline{\nu_1}) \leq -\overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g}.$$

Proposition 5. *If $B \geq 3$, then*

$$\sigma(\sigma - 2 - \overline{\nu_1}) \leq \nu_1 A_1 + 2\overline{g} - \overline{\nu_1} k - \tilde{k}. \tag{8}$$

Introducing an invariant λ^* by $\lambda^* = k - A_1$, we obtain

$$0 \leq \mathcal{Z}^* = B_2 \sigma(2 + \overline{\nu_1} - \sigma) - \lambda^* \overline{\nu_1} + A_1 + 2\overline{g} - \tilde{k}. \tag{9}$$

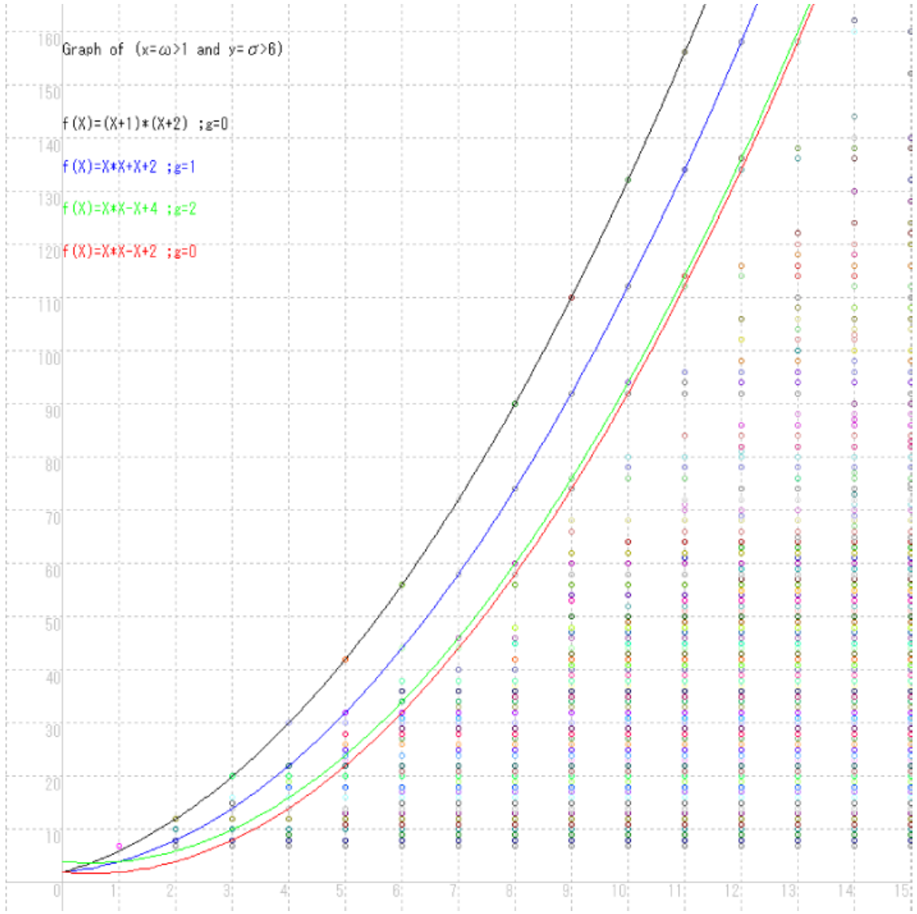


FIGURE 10

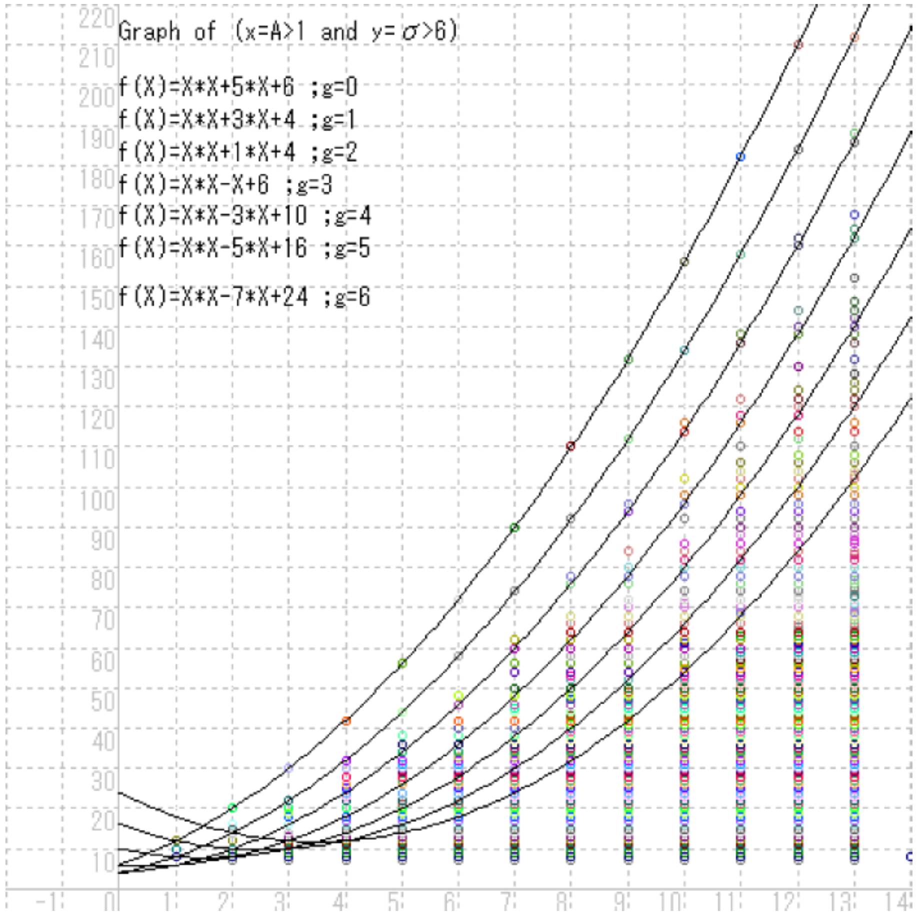

Graph of (x=A>1 and y=σ>6)
f(X)=X*X+5*X+6 ;g=0
f(X)=X*X+3*X+4 ;g=1
f(X)=X*X+1*X+4 ;g=2
f(X)=X*X-X+6 ;g=3
f(X)=X*X-3*X+10 ;g=4
f(X)=X*X-5*X+16 ;g=5
f(X)=X*X-7*X+24 ;g=6

FIGURE 11

Proposition 6. *If $B \leq 2$, then*

$$\lambda^* \overline{\nu_1} \leq A_1 + 2\overline{g} - \tilde{k}. \tag{10}$$

5.1. **lemma.**

Lemma 3. *Suppose that (S, D) is minimal.*

(1) *If $B = 0$ or 2 then*

$$\sigma A_1 + 4\overline{g} = \sigma Z^2 - 2(\sigma - 2)\overline{g} = (2D + \sigma K_S) \cdot Z \geq 0.$$

(2) *If $B = 1$ then either $\sigma A_1 + 4\overline{g} \geq 0$ or*

$$eA_1 + 6\overline{g} = (3D + eK_S) \cdot Z \geq 0.$$

(3) *If $B \geq 2$ then $\sigma A_1 + 4\overline{g} \geq \sigma(\sigma - 2)(B - 2)$.*

(4) *If $B \geq 3$ then $\sigma A_1 + 4\overline{g} \geq \sigma(\sigma - 2)$; in particular,*

$\sigma A_1 + 4\overline{g} = \sigma Z^2 - 2(\sigma - 2)\overline{g} \geq \sigma(\sigma - 2).$

5.2. **lower estimate of A.** Assuming $B \geq 3$, we shall obtain the lower estimate of A.

Proposition 7. *If $B \geq 3$ and $\nu_1 \geq 4$ then $A \geq 7$.*

If $A = 7$ then the type is either $[8*24, 3; 4^{12}, 3, 2^{\delta}]$ or $[8*25, 3; 4^{14}]$ where $\delta = 0, 1, 2$.

Proof. omitted.

6. ESTIMATE OF e IN TERMS OF A

Replacing σ by $\frac{e-u}{B}$, we get

$$2e - B\sigma - 2\sigma = e(1 - \frac{2}{B}) + u(1 + \frac{2}{B})$$

and then

$$\begin{aligned} \sigma A_1 + 4\overline{g} &= \sigma A - (\sigma - 4)\overline{g} \\ &\geq (2e - B\sigma - 2\sigma)(\sigma - 2) \\ &= e(1 - \frac{2}{B})(\sigma - 2) + u(1 + \frac{2}{B})(\sigma - 2). \end{aligned}$$

Hence, supposing that $\sigma \geq 5$, we get

$$\sigma A - (\sigma - 4)\overline{g} \geq e(1 - \frac{2}{B})(\sigma - 2) + u(1 + \frac{2}{B})(\sigma - 2). \tag{11}$$

Thus, when $g > 0$,

$$\sigma A \geq e(1 - \frac{2}{B})(\sigma - 2); \tag{12}$$

In other words,

$$e \leq \frac{B\sigma}{(B-2)(\sigma-2)}. \tag{13}$$

6.1. **case when $g = 0$.**

Next, we shall show the inequality : $e \leq \frac{B\sigma}{(B-2)(\sigma-2)}$ even if $g = 0$.

First assume that $g = 0$ and $e > \frac{B\sigma}{(B-2)(\sigma-2)}A$.

From the inequality

$$\sigma A \geq e(1 - \frac{2}{B})(\sigma - 2) + (1 + \frac{2}{B})u(\sigma - 2) + \sigma - 4\overline{g}, \tag{14}$$

It follows that

$$e\frac{(B-2)(\sigma-2)}{B} > \sigma A \geq e(1-\frac{2}{B})(\sigma-2)+(1+\frac{2}{B})u(\sigma-2)+\sigma-4\overline{g}.$$

Since $g = 0$,it follows that

$$0 \geq (1 + \frac{2}{B})u(\sigma - 2) + (\sigma - 4\overline{g})(\sigma - 2) = (1 + \frac{2}{B})u(\sigma - 2) + 4 - \sigma.$$

Hence,

$$1 > \frac{\sigma - 4}{(\sigma - 2)} > u + \frac{2u}{B} \geq u \geq 0. \tag{15}$$

Therefore, $u = 0$ and so $e = B\sigma$. Hence,

$$\sigma A \geq e(1 - \frac{2}{B})(\sigma - 2) + 4 - \sigma = (B - 2)\sigma(\sigma - 2) + 4 - \sigma.$$

In other words,

$$A \geq (B-2)(\sigma-2) + \frac{4}{\sigma} - 1.$$

By hypothesis, $-1 < \frac{4}{\sigma} - 1 \leq 0$.

Hence, by $e = B\sigma$, we get

$$A \geq (B-2)(\sigma-2) = e\frac{(\sigma-2)(B-2)}{B\sigma}.$$

Thus,

$$A \geq (B-2)(\sigma-2) = e\frac{(\sigma-2)(B-2)}{B\sigma}.$$

Consequently,

$$e \leq \frac{B\sigma}{(B-2)(\sigma-2)}A. \tag{16}$$

If $B \geq 4$ and $\sigma \geq 6$, then

$$e \leq \frac{B\sigma}{(B-2)(\sigma-2)}A \leq \frac{4}{2} \times \frac{6}{4}A = 3A. \tag{17}$$

Thus we obtain the next estimate.

Proposition 8. *If* $B \geq 4$ *and* $\sigma \geq 6$*, then*

$$e \leq 3A. \tag{18}$$

Next, we shall establish that even if $B \geq 3$,then $e \leq 3A$ except for certain cases.

7. NONBIRATIONAL INVARIANT e

Proposition 9. *If* $g = 0, \sigma \geq 7$ *and* $B \geq 3$ *then* $e < \frac{B}{B-2}\omega$.

Therefore, if $B \geq 3$,then

$$e < \frac{B}{B-2}\omega \leq 3\omega.$$

Assume $B \leq 2$.

1) $B = 0$.

$$e < \omega_1^2 + \omega_1 + 2 + 2\overline{g}. \tag{19}$$

$$e < \omega^2 + 3\omega + 2. \tag{20}$$

2) $B = 1$.

$$e < \frac{3}{2}(\omega_1^2 + \omega_1 + 2 + 2\overline{g}). \tag{21}$$

$$e < \frac{3}{2}(\omega^2 + 3\omega + 2). \tag{22}$$

3) $B = 2$.

$$e < 2(\omega_1^2 + \omega_1 + 2 + 2\overline{g}). \tag{23}$$

$$e < 2(\omega^2 + 3\omega + 2). \tag{24}$$

7.1. **case when $e > 3A$.**

In that follows, we assume $e > 3A$ and $\sigma \geq 6$.

By Proposition 8, we may assume that $B = 3$. Then

$$0 \leq \mathcal{Z}^* = \sigma(2 + \overline{\nu_1} - \sigma) - \overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g}.$$

Recalling that $\sigma = p + 2\overline{\nu_1} + 2$, we get

$$2 + \overline{\nu_1} - \sigma = 2 + \overline{\nu_1} - (p + 2\overline{\nu_1} + 2) = -p - \overline{\nu_1}.$$

Hence,

$$\begin{aligned}\mathcal{Z}^* &= -\sigma(p+\overline{\nu_1}) - \overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g}\\ &= \overline{\nu_1}(-\sigma - k + A_1) - \tilde{k} - p\sigma + A_1 + 2\overline{g}\\ &= \overline{\nu_1}(-\sigma - k + A) - \tilde{k} - p\sigma + A + \overline{g}(1-\overline{\nu_1}).\end{aligned}$$

But $A - \sigma < \frac{u}{3}$ and $-\sigma - k + A < \frac{u}{3} - k = -4p - \frac{5u}{3}$. Therefore, since $A < \sigma + \frac{u}{3} = p + \nu_1 + \frac{u}{3}$, it follows that

$$\begin{aligned}0 \le \mathcal{Z}^* &< -\overline{\nu_1}(4p + \frac{5u}{3}) - \tilde{k} - p\sigma + A + \overline{g}(1-\overline{\nu_1})\\ &< -\overline{\nu_1}(4p + \frac{5u}{3} - 2) - \tilde{k} - p\sigma + \frac{u}{3} + p + 2 + \overline{g}(1-\overline{\nu_1}).\end{aligned}$$

Thus,

$$0 \leq \overline{\nu_1}(4p + \frac{5u}{3} - 2) < -\tilde{k} + \frac{u}{3} + p + 2 - p\sigma + \overline{g}(1 - \overline{\nu_1}).$$

Supposing that $p > 0$, we shall derive a contradiction.
$4p + \frac{5u}{3} - 2 > 0$ and

$$\begin{aligned} -\tilde{k} + \frac{u}{3} + p + 2 &= -p(k - 2p) + \frac{u}{3} + p + 2 \\ &= -p(2p + 2u) + \frac{u}{3} + p + 2 \\ &= -2p^2 + 2 + p - pu - pu + \frac{u}{3} < 0, \end{aligned}$$

except for $p = 1$ and $u = 0$.

7.2. **case in which $p = 1$ and $u = 0$.** However, if $p = 1$ and $u = 0$ then $\sigma = 1 + 2\nu_1, k = 4, \tilde{k} = 1, A \leq 2\sigma - 1 = 2\nu_1$. Thus,

$$\begin{aligned} \mathcal{Z}^* &= \overline{\nu_1}(-\sigma - k + A) - \tilde{k} - p\sigma + A + \overline{g}(1 - \overline{\nu_1}) \\ &= \overline{\nu_1}(-1 - 2\nu_1 - 4 + A) - 1 - (1 + 2\nu_1) + A + \overline{g}(1 - \overline{\nu_1}) \\ &\leq -5\overline{\nu_1} - 2 + \overline{g}(1 - \overline{\nu_1}). \end{aligned}$$

But, if $\overline{g} \geq 0$, then

$$-5\overline{\nu_1} - 2 + \overline{g}(1 - \overline{\nu_1}) \leq -5\overline{\nu_1} - 2 + (1 - \overline{\nu_1}).$$

$$0 \leq \overline{\nu_1}(4p + \frac{5u}{3} - 2) < 0.$$

This is a contradiction.

By $p = 0$, we have

$$\begin{aligned}
\mathcal{Z}^* &= -\sigma(p + \overline{\nu_1}) - \overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g} \\
&= -2\nu_1(\overline{\nu_1}) - 2\overline{\nu_1}u + \nu_1 A_1 + 2\overline{g} \\
&= -2\overline{\nu_1}(u + \nu_1) + \nu_1 A_1 + 2\overline{g} \\
&= -2\nu_1(\overline{\nu_1} + u) + \nu_1 A_1 + 2\overline{g}.
\end{aligned}$$

From the inequality (11), we have

$$\sigma A \geq e(1-\frac{2}{B})(\sigma-2)+u(1+\frac{2}{B})(\sigma-2)+(\sigma-4)\overline{g}$$
$$=\frac{e(\sigma-2)}{3}+\frac{5u(\sigma-2)}{3}+(\sigma-4)\overline{g}.$$

Thus

$$3\sigma A \geq e(\sigma-2)+5u(\sigma-2)+3(\sigma-4)\overline{g}. \tag{25}$$

Since $e > 3\sigma$ by hypothesis, it follows that

$$e\sigma \geq e(\sigma - 2) + 5u(\sigma - 2) + 3(\sigma - 4)\overline{g}.$$

Hence,

$$2e \geq 5u(\sigma - 2) + 3(\sigma - 4)\overline{g}.$$

Since $p = 0$ we obtain

$$\begin{aligned} 2e = 2(3\sigma + u) &\geq 5u(\sigma - 2) + 3(\sigma - 4)\overline{g} \\ &= 10u(\nu_1 - 1) + 6(\nu_1 - 2)\overline{g}. \end{aligned}$$

Hence,

$$2(6\nu_1 + u) \geq 10u(\nu_1 - 1) + 6(\nu_1 - 2)\overline{g}.$$

Thus,

$$12u \geq 10u\nu_1 + 6\nu_1\overline{g} - 12\overline{g} - 12\nu_1;$$
$$6u + 6\nu_1 \geq 5u\nu_1 + 3\nu_1\overline{g} - 6\overline{g}.$$

Therefore,

$$6u + 6\overline{g} \geq 5u\nu_1 + 3\nu_1\overline{g} - 6\nu_1 = (5u + 3\overline{g} - 6)\nu_1. \tag{26}$$

Thus we have the next two cases to examine, separately.

1) $5u + 3\overline{g} - 6 < 0$. Then $u = 0, 1, 2$.
Suppose that $u = 1$. Then

$$\overline{g} \leq 2 - \frac{5u}{3} = \frac{1}{3} < 1.$$

Hence, $\overline{g} = 0, -1$.

(i) $\overline{g} = 0$.
Then

$$\begin{aligned} \mathcal{Z}^* &= -2\nu_1(\overline{\nu_1} + u) + \nu_1 A_1 + 2\overline{g} + 2u \\ &= 2\nu_1(1 - \nu_1) - 2(\nu_1 - 1) + \nu_1 A_1. \end{aligned}$$

Since $A \leq \sigma - 1 = 2\nu_1 - 1$, it follows that

$2\nu_1(1-\nu_1) - 2(\nu_1 - 1) + \nu_1 A_1 \leq 2\nu_1(1-\nu_1) - 2(\nu_1 - 1) < 0.$

Thus $\mathcal{Z}^* < 0$, a contradiction.

(ii) $\overline{g} = -1$. Then $A_1 = A + 1 \leq 2\nu_1$ and so

$$\begin{aligned}
\mathcal{Z}^* &= -2\nu_1(\overline{\nu_1} + u) + \nu_1 A_1 + 2\overline{g} \\
&= -2\nu_1(2\nu_1 - 1) + \nu_1(A_1) - 2 \\
&\leq -2\nu_1(2\nu_1 - 1) + \nu_1(2\nu_1) - 2 \\
&= 0.
\end{aligned}$$

Thus, $\mathcal{Z}^* = 0$. Hence,$e = 3\sigma + 1 = 6\nu_1 + 1$, $\widetilde{B} = 2e - 3\sigma = 3\sigma + 2$. Since $A = 2\nu_1 - 1$,it follows that $e - 3A = 4 > 0$.

$$g = 3(2\nu_1 - 1)\nu_1 - r\frac{\nu_1(\nu_1 - 1)}{2} = 0.$$

Therefore,

$$r = \frac{6(2\nu_1 - 1)}{\nu_1 - 1} = 12 + \frac{6}{\nu_1 - 1}. \tag{27}$$

From this, it follows that $\nu_1 - 1 = 2, 3, 6$ and we obtain the following types:

(1) The type is $[6 * 19, 3; 3^{15}]$, $A = 5$.
(2) The type is $[8 * 25, 3; 4^{14}]$, $A = 7$.
(3) The type is $[12 * 37, 3; 7^{13}]$, $A = 11$.

Suppose that $u = 2$. Then

$$\overline{g} \leq 2 - \frac{5u}{3} = \frac{-4}{3} < 1.$$

Hence, $\overline{g} < -1$, a contradiction.

Therefore, $u = 0$.

2) $5u + 3\overline{g} - 6 \geq 0$.

Then since $\sigma = 2\nu_1 \geq 6$, it follows that $\nu_1 \geq 3$ and thus

$$6u + 6\overline{g} \geq (5u + 3\overline{g} - 6)\nu_1 \geq 3(5u + 3\overline{g} - 6). \tag{28}$$

This implies the next inequality:

$$6 \geq 3u + \overline{g} \geq 3u - 1.$$

Hence, $u \leq 2$ and $\overline{g} \leq 6 - 3u$.

Moreover,

$$
\begin{aligned}
\mathcal{Z}^* &= -2\nu_1(\overline{\nu_1} + u) + \nu_1 A_1 + 2\overline{g} + 2u \\
&= -\nu_1(2(\overline{\nu_1} + u) - A_1) + 2\overline{g} + 2u.
\end{aligned}
$$

Then $A_1 = A - \overline{g} \leq 2\nu_1 - 1 - \overline{g}$ and so

$$
2(\overline{\nu_1} + u) - A_1 \geq 2(\overline{\nu_1} + u) - 2\nu_1 + 1 + \overline{g} = 2u - 1 + \overline{g}.
$$

Hence,

$$
\begin{aligned}
\mathcal{Z}^* &= -\nu_1(2(\overline{\nu_1} + u) - A_1) + 2\overline{g} + 2u \\
&\leq -\nu_1(2u - 1 + \overline{g}) + 2\overline{g} + 2u \\
&= -\nu_1(2u - 1) + 2u - \nu_1\overline{g} + 2\overline{g} \\
&< 0.
\end{aligned}
$$

This is absurd. Thus, $k = 0$ is proved.

7.3. **case when $k = 0$.**

$\varepsilon = 2\nu_1 - 1 - A$ satisfies $\varepsilon \geq 0$, since $A \leq \sigma - 1 = 2\nu_1 - 1$. From

$$0 \leq \mathcal{Z}^* = \nu_1(2 + A - 2\nu_1 - \overline{g}) + 2\overline{g}, \tag{29}$$

it follows that

$$\begin{aligned} \mathcal{Z}^* &= \nu_1(1 - \varepsilon - \overline{g}) + 2\overline{g} \\ &= \nu_1(1 - \varepsilon) + \overline{g}(2 - \nu_1). \end{aligned}$$

If $\overline{g} \geq 0$, then

$$\mathcal{Z}^* = \nu_1(1 - \varepsilon) + \overline{g}(2 - \nu_1) \leq \nu_1(1 - \varepsilon).$$

Hence, $\varepsilon \leq 1$.

Otherwise,

$$\mathcal{Z}^* = \nu_1(1 - \varepsilon) - (2 - \nu_1) = \nu_1(2 - \varepsilon) - 2.$$

Hence, $\varepsilon \leq 1$.

Thus we have the following cases:

(1) $\varepsilon = 0, \mathcal{Z}^* = \nu_1 + \overline{g}(2 - \nu_1)$

(a) $\overline{g} = -1, \mathcal{Z}^* = 2\nu_1 - 2,$

(b) $\overline{g} = 0, \mathcal{Z}^* = \nu_1,$

(c) $\overline{g} = 1, \mathcal{Z}^* = 2,$

(d) $\overline{g} = 2, \mathcal{Z}^* = 4 - \nu_1,$

(e) $\overline{g} = 3, \mathcal{Z}^* = 2(3 - \nu_1).$

(2) $\varepsilon = 1, \mathcal{Z}^* = \overline{g}(2 - \nu_1)$

(a) $\overline{g} = -1, \mathcal{Z}^* = \nu_1 - 2,$

(b) $\overline{g} = 0, \mathcal{Z}^* = 0,$

(c) $\overline{g} = 1, \mathcal{Z}^* = 2 - \nu_1.$

7.4. **case when** $\varepsilon = 0, \overline{g} = -1$**.**

In this case, $A = 2\nu_1 - 1, g = 0$.

If $\nu_1 = 3$,then $\mathcal{Z}^* = t_2$ and so $t_2 = 4$. The type is $[6 * 18, 3; 3^{t_3}, 2^4]$.

$g = 5 \cdot 8 - 3t_3 - 4 = 0$. Hence, $t_3 = 12$ and the type becomes $[6 * 18, 3; 3^{12}, 2^4]$ and $Z^2 = 4$.

If $\nu_1 = 4$,then $\mathcal{Z}^* = 2(t_2 + t_3)$ and so $t_2 + t_3 = 3$. The type is $[8 * 24, 3; 4^{t_4}, 3^{t_3}, 2^{t_2}]$.

$g = 7 \cdot 11 - 6t_4 - 3t_3 - t_2 = 0$. Hence, $37 = 3t_4 + t_3$.Then $t_3 = 1, t_4 = 12$ and the type becomes $[8 * 24, 3; 4^{12}, 3, 2^2]$. $Z^2 = 6$.

If $\nu_1 > 4$,then $\mathcal{Z}^* > \nu_1 - 2 + 2(\nu_1 - 3)$. From $\mathcal{Z}^* = 2\nu_1 - 2$, it follows that $\nu_1 \leq 6$. Hence,$\nu_1 = 5$ or 6.

If $\nu_1 = 5$,then $\mathcal{Z}^* = 3(t_2+t_4)+4t_3$ and so $3(t_2+t_4)+4t_3 = 8$. Then $t_3 = 2$ and the type is $[10*30, 3; 5^{t_5}, 3^2]$. $g = 126-(10t_5+6) = 0$. From this it follows that $t_5 = 12$ and the type becomes $[10 * 30, 3; 5^{12}, 3^2]$.

If $\nu_1 = 6$,then $\mathcal{Z}^* = 4(t_2 + t_5) + 6(t_3 + t_4) = 10$ and so $t_2 + t_5 = t_3 + t_4 = 1$. Then the type is $[12 * 36, 3; 6^{t_6}, 5^{t_5}, 4^{t_4}, 3^{t_3}, 2^{t_2}]$.

- $g = 187 - (15t_6 + 10t_5 + 6t_4 + 3t_3 + t_2) = 0$,
- $t_2 + t_5 = t_3 + t_4 = 1$.

These imply $t_2 = 1, t_4 = 1, t_6 = 12$. The type is $[12 * 36, 3; 6^{12}, 4, 2]$.

7.5. **case when** $\varepsilon = 0, \overline{g} = 0$.

In this case, $A = 2\nu_1 - 1, g = 1$ and $\mathcal{Z}^* = \nu_1$. Then $\nu_1 = \mathcal{Z}^* \geq 2(\nu - 3)$. Hence, $\nu_1 \leq 6$.

We have the four cases to examine, separately.

(1) $\nu_1 = 3$. Then $\mathcal{Z}^* = t_2 = 3$. The type is $[6 * 18, 3; 3^{t_3}, 2^3]$.

$g = 5 \cdot 8 - 3t_3 - 3 = 1$. Hence, $t_3 = 12$ and the type becomes $[6 * 18, 3; 3^{12}, 2^3]$ and $Z^2 = 5$.

(2) $\nu_1 = 4$. Then $\mathcal{Z}^* = 2(t_2 + t_3) = 4$ and so $t_2 + t_3 = 2$. The type is $[8 * 24, 3; 4^{t_4}, 3^{t_3}, 2^{t_2}]$.

$g = 7 \cdot 11 - 6t_4 - 3t_3 - t_2 = 1$. Hence, $37 = 3t_4 + t_3$. Then $t_3 = 1, t_4 = 12$ and the type becomes $[8 * 24, 3; 4^{12}, 3, 2]$. $Z^2 = 6 \cdot 20 - 9 \cdot 12 - 4 - 1 = 7, A = Z^2 = 7$.

(3) $\nu_1 = 5$. Then $\mathcal{Z}^* = 3(t_2 + t_4) + 4t_3 = 4$ and so $t_3 = 1, t_2 = t_4 = 0$. The type is $[10 * 30, 3; 5^{t_5}, 3]$.

$g = 9 \cdot 14 - 10t_5 - 3 = 1$, which has no solution.

(4) $\nu_1 = 6$. Then $\mathcal{Z}^* = 4(t_2 + t_5) + 6(t_3 + t_4) = 6$ and so $t_3 + t_4 = 1, t_2 = t_5 = 0$. The type is $[12 * 36, 3; 6^{t_6}, 4^{t_4}, 3^{t_3}]$.

- $g = 187 - (6t_4 + 3t_3) = 1$,
- $t_3 + t_4 = 1$.

These imply $t_4 = 1, t_6 = 12$. The type is $[12 * 36, 3; 6^{12}, 4]$.

7.6. **case when** $\varepsilon = 0, \overline{g} = 1$.

In this case, $A = 2\nu_1 - 1, g = 2$ and $\mathcal{Z}^* = 2$. From $\mathcal{Z}^* \geq nu_1 - 2$, it follows that $\nu_1 = 3$ or 4.

If $\nu_1 = 3$,then $\mathcal{Z}^* = t_2 = 2$. The type is $[6 * 18, 3; 3^{t_3}, 2^2]$.

$g = 5 \cdot 8 - 3t_3 - 3 = 2$. Hence, $t_3 = 12$ and the type becomes $[6 * 18, 3; 3^{12}, 2^2]$ and $Z^2 = 5$.

If $\nu_1 = 4$,then $\mathcal{Z}^* = 2(t_2 + t_3) = 2$ and so $t_2 + t_3 = 1$. The type is $[8 * 24, 3; 4^t_4, 3^{t_3}, 2^{t_2}]$.

$g = 7 \cdot 11 - 6t_4 - 3t_3 - t_2 = 2$. Hence, $37 = 3t_4 + t_3$.Then $t_3 = 1, t_4 = 12$ and the type becomes $[8 * 24, 3; 4^{12}, 3]$

7.7. **case when** $\varepsilon = 0, \overline{g} = 2$**.**

In this case, $A = 2\nu_1 - 1, g = 3$ and $\mathcal{Z}^* = 4 - \nu_1$.

If $4 \neq \nu_1$ then $4 - \nu_1 \geq \nu_1 - 2$; thus $\nu_1 = 3$.

If $\nu_1 = 4$,then $\mathcal{Z}^* = 0$ and so $t_3 = 0$. The type is $[8 * 24, 3; 4^{t_4}]$. $g = 7 \cdot 11 - 6t_4 = 3$, which has no solution.

If $\nu_1 = 3$,then $\mathcal{Z}^* = 1$. Then $t_2 = 1$ and The type is $[6 * 18, 3; 3^{t_3}, 2]$.

$g = 5 \cdot 8 - 3t_3 - 3 = 3$. Hence, $t_3 = 12$ and the type becomes $[6 * 18, 3; 3^{12}, 2]$ and $Z^2 = 6$.

7.8. **case when** $\varepsilon = 0, \overline{g} = 3$**.**

In this case, $A = 2\nu_1 - 1, g = 4$ and $\mathcal{Z}^* = 2(3 - \nu_1)$.

Then $\nu_1 = 3$ and the type becomes $[6 * 18, 3; 3^{12}]$ and $Z^2 = 7$.

7.9. **case when** $\varepsilon = 1, \overline{g} = -1$**.**

In this case, $A = 2\nu_1 - 2, g = 0$ and $\mathcal{Z}^* = \nu_1 - 2$.

(1) $\nu_1 = 3$. Then $t_2 = 1$ and $g = 40 - 3t_3 - 1 = 0$. Hence, $t_3 = 13$ and the type is $[6 * 18; 3^{13}, 2]$. In this case, $Z^2 = 3, A = 4$. Here, $18 > 3A = 12$.

(2) $\nu_1 > 3$.

Since $\mathcal{Z}^* = \nu_1 - 2$, it follows that $t_2 + t_{\nu_1 - 1} = 1$.

Note that by $\nu_1 > 3$ we have

$$g = (2\nu_1 - 1)(3\nu_1 - 1) - \frac{\nu_1(\nu_1 - 1)}{2} t_{\nu_1} - t_2 - \frac{(\nu_1 - 2)(\nu_1 - 1)}{2} t_{\nu_1 - 1} = 0.$$

If $t_2 = 1$ then $t_{\nu_1 - 1} = 0$ and hence,

$$12\nu_1 - 10 - (\nu_1 - 1)t_{\nu_1} = 0.$$

Thus,

$$t_{\nu_1} = \frac{12\nu_1 - 10}{\nu_1 - 1} = 12 + \frac{2}{\nu_1 - 1}.$$

Therefore, $\nu_1 = 3$, a contradiction.

If $t_2 = 0$ then $t_{\nu_1 - 1} = 1$ and hence,

$$g = (2\nu_1 - 1)(3\nu_1 - 1) - \frac{\nu_1(\nu_1 - 1)}{2}t_{\nu_1} - \frac{(\nu_1 - 2)(\nu_1 - 1)}{2} = 0.$$

$$11\nu_1 - 7 - (\nu_1 - 1)t_{\nu_1} = 0.$$

Thus,

$$t_{\nu_1} = \frac{11\nu_1 - 7}{\nu_1 - 1} = 11 + \frac{4}{\nu_1 - 1}.$$

Therefore, $\nu_1 = 5$, because, $\nu_1 > 3$,$g = 126 - (10t_5 + 6) = 0$. The type becomes $[10 * 30, 3; 5^{12}, 4]$.

If $t_2 = 1$ then $t_{\nu_1 - 1} = 0$ and hence,

$$12\nu_1 - 10 - (\nu_1 - 1)t_{\nu_1} = 0.$$

Thus,

$$t_{\nu_1} = \frac{12\nu_1 - 10}{\nu_1 - 1} = 12 + \frac{2}{\nu_1 - 1}.$$

Therefore, $\nu_1 = 3$ and $t_3 = 13$. $g = 5 \cdot 8 - 3 \times 13 - t_2 = 0$. Hence, $t_2 = 1$ and the type becomes $[6 * 18, 3; 3^{12}, 2]$ and $Z^2 = 3$. Note that $A = 3 + 1 = 4 = 2\nu_1 - 2$.

If $t_2 = 0$ then $t_{\nu_1 - 1} = 1$ and hence,

$$g = (2\nu_1 - 1)(3\nu_1 - 1) - \frac{\nu_1(\nu_1 - 1)}{2} t_{\nu_1} - \frac{(\nu_1 - 2)(\nu_1 - 1)}{2} = 0.$$

$$11\nu_1 - 7 - (\nu_1 - 1)t_{\nu_1 - 1} = 0.$$

Thus,

$$t_{\nu_1} = \frac{11\nu_1 - 7}{\nu_1 - 1} = 11 + \frac{4}{\nu_1 - 1}.$$

Therefore, $\nu_1 = 5$ and $t_5 = 12, t_4 = 1$. Then $g = 126 - (10t_5 + 6) = 0$. The type becomes $[10 * 30, 3; 5^{12}, 4]$.

7.10. **case when** $\varepsilon = 1, \overline{g} = 0$**.**

In this case, $A = 2\nu_1 - 2, g = 1$ and $\mathcal{Z}^* = 0$.

It follows that

$$g = (2\nu_1 - 1)(3\nu_1 - 1) - \frac{\nu_1(\nu_1 - 1)}{2} t_{\nu_1} = 1.$$

Hence,

$$12\nu_1 - 10 - (\nu_1 - 1)t_{\nu_1} = 0.$$

Thus,

$$t_{\nu_1} = \frac{12\nu_1 - 10}{\nu_1 - 1} = 12 + \frac{3}{\nu_1 - 1}.$$

Therefore, $\nu_1 = 4$ and $t_3 = 0$. The type is $[8 * 24, 3; 4^{t_4}]$. $g = 7 \cdot 11 - 6t_4 = 1$, which has no solution.

7.11. **case when** $\varepsilon = 1, \overline{g} = 1$**.**

But this case does not occur.

Therefore, we obtain the following

Theorem 5. *If $B \geq 3$ and $\sigma > 4$, then $e \leq 3A$ except for the following cases:*

(1) $u = 1$,

(a) *The type is* $[6 * 19, 3; 3^{15}]$, $A = 5$.

(b) *The type is* $[8 * 25, 3; 4^{14}]$, $A = 7$.

(c) *The type is* $[12 * 37, 3; 7^{13}]$, $A = 11$.

(2) $u = 0$,

(a) *The type is* $[6 * 18, 3; 3^{12}, 2^{\delta}]$, $\delta \leq 4$,$A = 2\nu_1 - 1 = 5$,

(b) *The type is* $[8 * 24, 3; 4^{12}, 3, 2^{\delta}]$, $\delta \leq 2$,,$A = 2\nu_1 - 1 = 7$,

(c) *The type is* $[10 * 30, 3; 5^{12}, 3^{2}]$,$A = 2\nu_1 - 1 = 9$,

(d) *The type is* $[12 * 36, 3; 6^{12}, 4, 2^{\delta}]$, $\delta \leq 1$,$A = 2\nu_1 - 1 = 11$,