Birational characterization of nonsingular plane curves

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1 Introduction

We shall study algebraic plane curves C on the projective plane $\mathbf{P^2}$ defined over the field of complex numbers. Birational maps between from $\mathbf{P^2}$ into itself are called Cremona transformations. If C_1 is a proper transform of C by a Cremona transformation, the pair $(\mathbf{P^2}, C_1)$ is said to be birationally equivalent to $(\mathbf{P^2}, C)$. The purpose of this paper is to give certain conditions which characterize $(\mathbf{P^2}, D)$ where D is a nonsingular curve, in the sense of birational equivalence.

In general, let C be a curve on a nonsingular projective surface S. Pairs (S, C) of S and C are objects of our study. Two pairs (S, C) and (S_1, C_1) are said to be birationally equivalent if there exists a birational map $h : S \to S_1$ such that the proper transform $h[C]$ coincides with C_1. If D is a nonsingular curve on S, then it is easy to check that $\dim |mK_S + aD| + 1$, K_S being a canonical divisor on S, are birational invariants whenever $m \geq a \geq 0$. $\dim |mK_S + aD| + 1$ are denoted by $P_{m,a}[D]$, which may be called mixed plurigenera of the pair (S, D). $P_{m,m}[D]$ turns out to be logarithmic plurigenera of an open surface $S - D$, denoted by $\overline{P_m}(S - D)$. For simplicity, $P_{m,m}[D]$ is indicated by $P_m[D]$, by which Kodaira dimension of the pair (S, C), written as $\kappa[C]$, is defined.

Hereafter,S is assumed to be a rational surface. Then $P_1[D]$ coincides with the genus of D, denoted by $g(D)$. Making use of mixed plurigenera, we obtain the characterizations of a line and a nonsingular cubic as follows:

Theorem 1 *Let (S, D) be a pair of a nonsingular projective surface S and a curve on S.*

If $P_{2,1}[D] = 0$ and $g(D) = 0$ then (S, D) is birationally equivalent to $(\mathbf{P^2}, L)$, L being a line.

Note that the condition $P_{2,1}[D] = 0$ and $g(D) = 0$ is equivalent to $P_2[D] = 0$.

Theorem 2 *If $P_{2,1}[D] = 1$ and $g(D) = 1$ then (S, D) is birationally equivalent to $(\mathbf{P^2}, C_3)$, C_3 being a nonsingular cubic.*

These results are mainly due to [1, p398,p404]. We shall extend his results into higher degree cases.

We begin with computing mixed plurigenera $P_{m,a}[D]$ when $(S, D) = (\mathbf{P^2}, C_d)$, C_d being a nonsingular curve of degree d. For $m \geq a$ and $d \geq 4$,

1. $P_{m,a}[D] = \dfrac{(3m - 1 - ad)(3m - 2 - ad)}{2}$,
2. $P_m[D] = \dfrac{((d - 3)m + 1)((d - 3)m + 2)}{2}$,
3. $P_1[D] = \dfrac{(d - 2)(d - 1)}{2} = g(D)$,
4. $P_2[D] = (d - 2)(2d - 5)$,

5. $P_{2,1}[D] = \dfrac{(d-4)(d-5)}{2}$,

6. $P_{3,1}[D] = \dfrac{(d-7)(d-8)}{2}$ where $d \geq 7$.

One can ask to what extent (S, D) is determined by its mixed plurigenera. Our purpose is to establish some characterizations of pairs of $\mathbf{P^2}$ and nonsingular curves using two mixed plurigenera, which will be established in main results. For examples, if $P_2[D] = 6$ and $g = 3$ then (S, D) is birationally equivalent to $(\mathbf{P^2}, C_4)$.

The similar results are obtained for $d = 6$. However, in the case of $d = 5$, we have a counter example:

If $P_2[D] = 10$ and $g = 6$ then (S, D) is birationally equivalent to either $(\mathbf{P^2}, C_5)$ or $(\mathbf{P^2}, C'_6)$, where C'_6 is a plane curve of degree 6 with two singular points whose multiplicities are 2 and 3.

2 Some basic results

2.1 Minimal models

A non-singular pair (S, D) is said to be *relatively minimal* , whenever the intersection number $D \cdot E \geq 2$ for any exceptional curve (of the first kind) E on S such that $E \neq D$. Moreover, the pair (S, D) is said to be *minimal* , if every birational map from any non-singular pair (S_1, D_1) into (S, D) turns out to be regular. Any relatively minimal pair (S, D) is minimal if $\kappa[D] = 2$ (see Iitaka [5]).

Relatively minimal models of rational surfaces are the projective plane $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$ or a $\mathbf{P}^1-$ bundle over $\mathbf{P}^1$, which has a section Δ_∞ with negative self intersection number. The last surface is denoted by a symbol Σ_B where $-B$ denotes the self intersection number ${\Delta_\infty}^2$. Here, we call Σ_B a Hirzebruch surface of degree B after Kodaira. The Picard group of Σ_B is generated by a section Δ_∞ and a fiber $F_c = \rho^{-1}(c)$ of the $\mathbf{P}^1-$ bundle, where $\rho : \Sigma_B \to \mathbf{P}^1$ is the projection.

Let C be an irreducible curve on Σ_B. Then there exist integers σ and e such that

$$C \sim \sigma\Delta_\infty + eF_c.$$

Here the symbol $\sim$ means the linear equivalence between divisors.

We have $C \cdot F_c = \sigma$ and $C \cdot \Delta_\infty = e - B \cdot \sigma$. Hereafter, suppose that $C \neq \Delta_\infty$. Thus $C \cdot \Delta_\infty \geq 0$ and hence, $e \geq B\sigma$. If $B > 0$ then ${\Delta_\infty}^2 = -B < 0$ and such a section Δ_∞ is uniquely determined. For a surface $\Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$, we get $F_c \sim \mathbf{P}^1\times$ point and $\Delta_\infty \sim$ point $\times \mathbf{P}^1$. We may assume that $e \geq \sigma$. Thus σ and e are uniquely determined for a given curve C on Σ_B.

By $\nu_1, \nu_2, \cdots, \nu_r$ we denote the multiplicities of singular points of C where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$.

The symbol $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$ is said to be the type of a pair (Σ_B, C). If B=0, we omit 0 in the symbol of type; namely, $[\sigma * e; \nu_1, \nu_2, \cdots, \nu_r]$ stands for $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$.

Assume that $\sigma \geq 2\nu_1$ and $e \geq \sigma + B\nu_1$. Moreover, if $B = 1$ then assume $e - \sigma > 1$. When the above conditions are satisfied, the pair (Σ_B, C) is said to be # *minimal.* Occasionally, the # minimal pair (Σ_B, C) is said to be a # *minimal model* of a pair (S, D), if it is birationally equivalent to (S, D) (See [5]). Moreover, any minimal pair (S, D) is obtained from a # minimal model by resolving singularities of C, if it is not isomorphic to $(\mathbf{P^2}, C_d)$, C_d being a nonsingular curve (See [5]).

If (S, D) is minimal and $\kappa[D] = 2$, then the following results are obtained(see [7]).

1. If $g \geq 1$ and $\sigma \geq 4$ then $P_2[D] = Z^2 + 2\overline{g} + 1$.
2. If $g \geq 0$ and $\sigma \geq 4$ then $P_{2,1}[D] = Z^2 - \overline{g} + 1$.
3. If $g \geq 0$, $\sigma \geq 6$ and the type is not $[6 * 8, 1; 2^r]$ for $r \geq 0$, then $P_{3,1}[D] = 3Z^2 - 7\overline{g} + D^2 + 1$.
4. If $g \geq 1$ then $P_2[D] = P_{2,1}[D] + 3\overline{g}$.
5. If $g = 0$ then $P_2[D] = P_{2,1}[D] = Z^2 + 2$.

Here $\overline{g} = g - 1$.

The next result may be noteworthy.

Remark 1 *If the pair (S, D) satisfies that $g(D) = \dfrac{(d-2)(d-1)}{2}, P_2[D] = (d-2)(2d-5), P_{3,1}[D] = \dfrac{(d-7)(d-8)}{2}$, then (S, D) is birationally equivalent to $(\mathbf{P^2}, C_d)$, where C_d is a nonsingular curve with degree d.*

In order to verify this, we can assume that (S, D) is minimal.

It is easy to check $\kappa[D] = 2$. Then $Z = K_S + D$ is nef and big. By the formulas $P_2[D] = Z^2 + 2\overline{g}, P_{3,1}[D] = 3Z^2 + D^2 - 7\overline{g}$, $\overline{g}$ being $g - 1$, the hypothesis implies that $D^2 = d^2, Z^2 = (d-3)^2$. From the formula $Z^2 = K_S^2 - D^2 + 4\overline{g}$, it follows that $(d-3)^2 = K_S^2 - d^2 + 2d(d-3)$. Hence, $K_S^2 = 9$. This yields that $S = \mathbf{P^2}$, which completes the proof.

This result suggests that giving values of three mixed plurigenera such as $g, P_2[D], P_{3,1}[D]$ is superabundant.

2.2 Formulas

Letting g_0 be the virtual genus of C , K_0 a canonical divisor on Σ_B and defining Z_0 to be $C + K_0$, we get

$$g_0 = (e-1)(\sigma-1) - \frac{B\sigma(\sigma-1)}{2},$$

$$C^2 = 2e\sigma - \sigma^2 B.$$

Moreover, letting $f = e - B\sigma = C \cdot \Delta_0 \geq 0$, we obtain

$$C \sim \sigma\Delta_0 + fF_c,$$

$$K_0 \sim -2\Delta_0 + (B-2)F_c,$$

$$Z_0 = C + K_0 \sim (\sigma-2)\Delta_0 + (f-2+B)F_c,$$

where Δ_0 is an irreducible curve linearly equivalent to $\Delta_\infty + BF_c$.

Denoting $2f + \sigma B$ by $\tilde{B}$, we find

$$g_0 = \frac{(\sigma-1)(\tilde{B}-2)}{2}, \quad C^2 = \sigma\tilde{B},$$

$$Z_0^2 = (\sigma-2)(\tilde{B}-4),$$

$$(2Z_0 - C)\cdot Z_0 = (\sigma-3)(\tilde{B}-6) - 2,$$

$$(2Z_0 - C)\cdot(3Z_0 - 2C) = (\sigma-5)(\tilde{B}-10) - 2.$$

These formulas suggest that $\tilde{B}$ is very useful. Hence, we introduce the following notion.

Two types $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$ and $[\sigma * e', B'; \nu_1, \nu_2, \cdots, \nu_r]$ are said to be **similar** if $\tilde{B} = \tilde{B}'$, where $f' = e' - \sigma B'$ and $\tilde{B}' = 2f' + \sigma B'$. For simplicity, we omit the similar types in the following tables of types of pairs.

2.3 virtual mixed plurigenera

If C is a curve on S, define $VP_{m,a}[C]$ to be $\dim |mK_S + aC| + 1$, which we call virtual mixed plurigenus of the pair (S, C).

Let (S, D) be a pair derived from a # minimal pair (Σ_B, C) of type $[\sigma * e, B; \nu_1, \cdots, \nu_r]$, by resolving singularities of C. Then by E_i denoting the exceptional divisor arising from the singular points p_j of C, we obtain

$$mK_S + aD \sim mK_0 + aC + \sum_{j=1}^{r}(m - a\nu_i)E_i.$$

Suppose that$m \geq a\nu_1$. Then

$$|mK_S + aD| = |mK_0 + aC| + \sum_{j=1}^{r}(m - a\nu_i)E_i.$$

Hence,

$$VP_{m,a}[C] = P_{m,a}[D].$$

Therefore, we obtain the next result.

Lemma 1 *Let (S, D) be a pair. If $m \geq a\nu_1$ then $VP_{m,a}[C] = P_{m,a}[D]$.*

Equivalently, the next result follows.
If $VP_{m,a}[C] > P_{m,a}[D]$ then $m < a\nu_1$

Note that this result implies the famous Noether's inequality in the theory of Cremonian geometry.

2.4 Hartshorne's lemma

The next result came from the proof in [2, Hartshorne,Proposition (3.2),p118].

Lemma 2 *Let (S, D) be a minimal pair derived from a # minimal pair (Σ_B, C) of type $[\sigma * e, B; \nu_1, \cdots, \nu_r]$, by resolving singularities of C. Then we have either (1) $|\sigma Z - (\sigma - 2)D| \neq \emptyset$ or (2) $B = 1, 2f < \sigma$ and $|eZ - (e-3)D| \neq \emptyset$.*

Proof. By E_i denoting the exceptional divisor arising from the singular points p_j of C, we obtain

$$\begin{aligned}
\sigma Z - (\sigma - 2)D &= \sigma K_S + 2D \\
&\sim 2(\sigma\Delta_0 + fF_c - \sum_{j=1}^{r}\nu_i E_i) \\
&+ \sigma(-2\Delta_0 + (B-2)F_c + \sum_{j=1}^{r} E_i) \\
&\sim (2f + \sigma(B-2))F_c + \sum_{j=1}^{r}(\sigma - 2\nu_i)E_i.
\end{aligned}$$

Letting ε_1 be $2f + \sigma(B-2)$, we have the following two cases:
(1) If $B = 0$ then $\varepsilon_1 = 2f - 2\sigma \geq 0$ and if $B \geq 2$ then $\varepsilon_1 \geq 0$.
(2) if $B = 1$ and if $\varepsilon_1 = 2f - \sigma < 0$ then $3\sigma - 2e = \sigma - 2f = -\varepsilon_1 > 0$ and hence, $|\sigma Z - (\sigma - 2)D| = \emptyset$. In this case,

$$e - 3\nu_i \geq e - 3\nu_1 \geq e - \nu_1 - 2\nu_1 \geq \sigma - 2\nu_1 \geq 0.$$

Thus,

$$\begin{aligned}
eZ-(e-3)D &= eK_S+3D \\
&\sim 3(\sigma\Delta_0+fF_c-\sum_{j=1}^{r}\nu_iE_i) \\
&\quad + e(-2\Delta_0+(B-2)F_c+\sum_{j=1}^{r}E_i) \\
&\sim (3\sigma-2e)(\Delta_0-F_c)+\sum_{j=1}^{r}(e-3\nu_i)E_i \\
&\sim (3\sigma-2e)\Delta_\infty+\sum_{j=1}^{r}(e-3\nu_i)E_i.
\end{aligned}$$

Therefore, $|eZ-(e-3)D| \neq \emptyset$, which completes the proof.

Note that $P_{\sigma,2}[D] = VP_{\sigma,2}[C] and P_{e,3}[D] = VP_{e,3}[C]$.

The next result follows from Lemma 1 immediately.

Lemma 3 *Let (S,D) be a minimal pair derived from a # minimal pair (Σ_B, C) of type $[\sigma * e, B; \nu_1, \cdots, \nu_r]$.*

1. *Either (1) $\sigma Z^2 \geq 2(\sigma-2)\overline{g}$ or (2) $B=1$ and $eZ^2 \geq 2\overline{g}(e-3)$.*
2. *Either (1) $2\sigma\overline{g} \geq (\sigma-2)D^2$ or (2) $B=1$ and $2\overline{g}e \geq (e-3)D^2$.*

Here g denotes the genus of D.

Proof. The assertion 1 follows from the fact that Z is nef where $g>0$. In order to verify (1) of the assertion 2, assume that

$$2\sigma\overline{g}-(\sigma-2)D^2 = (\sigma Z-(\sigma-2)D)\cdot D < 0.$$

Then since $|\sigma Z-(\sigma-2)D)| \neq \emptyset$, it follows that $D^2<0$ and $2\sigma\overline{g} < (\sigma-2)D^2 \leq 0$. Hence, $g=0$. Then noting that $\sigma \geq 4$, we have

$$-2-\frac{4}{\sigma-2} \geq -4 \text{ and } -4 \geq D^2$$

and thus

$$-2-\frac{4}{\sigma-2} \geq D^2.$$

It follows that $2\sigma\overline{g} = -2\sigma \geq (\sigma-2)D^2$.

By the similar argument, we are done in the assertion 2.

3 Bigenus and genus

Suppose that (S, D) is a minimal pair which satisfies (1) $P_2[D] = (2d-5)(d-2)$, for some $d \geq 4$ and (2) $\delta = g - \frac{(d-1)(d-2)}{2} \geq 0$, g being the genus of D.

Assume that (S, D) is *not* birationally equivalent to $(\mathbf{P^2}, C_d)$,C_d being a nonsingular curve. Then (S, D) is obtained from a # minimal model (Σ_B, C) with type $[\sigma * e, B; \nu_1, \cdots, \nu_r]$ by shortest resolution of singularities of C. From the formula $P_2[D] = Z^2 + 2g - 1$, Z being $K_S + D$ ([7]), it follows that

$$(2d-5)(d-2) = Z^2 + 2g - 1 = Z^2 + d^2 - 3d + 2 + 2\delta - 1.$$

Hence ,

$$Z^2 = (d-3)^2 - 2\delta. \tag{1}$$

Denoting by t_j the numbers of singular points of C with multiplicities j, define X to be $\sum_{j=2}^{\nu_1} \frac{j(j-1)}{2} t_j$. Then by genus formula,

$$(\sigma - 1)(\tilde{B} - 2) = 2g + 2X = d^2 - 3d + 2 + 2\delta + 2X. \tag{2}$$

Moreover, defining U to be $\sum_{j=2}^{\nu_1} (j-1)^2 t_j$, we get

$$Z^2 + U = (\sigma - 2)(\tilde{B} - 4). \tag{3}$$

Multiplying (3) by $\sigma - 1$, we have

$$\begin{aligned}(\sigma-1)Z^2 + (\sigma-1)U &= (\sigma-2)((\sigma-1)(\tilde{B}-2) - 2(\sigma-1)) \\ &= (\sigma-2)(2g + 2X - 2(\sigma-1)) \\ &= (\sigma-2)(d^2 - 3d + 2 + 2\delta) + 2(\sigma-2)X - 2(\sigma-1)(\sigma-2).\end{aligned}$$

On the other hand,

$$(\sigma-1)Z^2 + (\sigma-1)U = (\sigma-1)((d-3)^2 - 2\delta) + (\sigma-1)U.$$

From these , it follows that

$$(\sigma-1)((d-3)^2 - 2\delta) - (\sigma-2)(d^2 - 3d + 2)$$

$$= 2\delta(\sigma-2) + 2(\sigma-2)X - (\sigma-1)U - 2(\sigma-1)(\sigma-2).$$

Defining Θ_2 to be $2(\sigma-2)X - (\sigma-1)U$, we have

$$\Theta_2 = \sum_{j=2}^{\nu_1} \{(\sigma-2)j(j-1) - (\sigma-1)(j-1)^2\} t_j$$

$$= \sum_{j=2}^{\nu_1} \{(j-1)(\sigma - j - 1)\} t_j,$$

and

$$(\sigma - 1)(d-3)^2 - (\sigma - 2)(d^2 - 3d + 2) + 2(\sigma - 1)(\sigma - 2)$$

$$= d^2 - 3\sigma d + 2\sigma^2 + \sigma - 1$$

$$= (d - \sigma - 1)(d - 2\sigma + 1).$$

Finally, we find the following formula:

$$(d - \sigma - 1)(d + 1 - 2\sigma) = 2(2\sigma - 3)\delta + \Theta_2, \tag{4}$$

where

$$\Theta_2 = \sum_{j=2}^{\nu_1} (j-1)(\sigma - j - 1) t_j = (\sigma - 3)t_2 + 2(\sigma - 4)t_3 + 3(\sigma - 5)t_4 + \cdots .$$

By $(d - \sigma - 1)(d + 1 - 2\sigma) \geq 0$, we have either $d \geq 2\sigma - 1$ or $d \leq \sigma + 1$.

3.1 Estimate of d

We shall show that $d \geq 2\sigma - 1$. First, by Lemma 2, we obtain either (1) $\sigma Z^2 \geq 2(\sigma - 2)\overline{g}$ or (2) $B = 1$ and $eZ^2 \geq 2(e - 3)\overline{g}$.

In the first case,

$$\sigma Z^2 = \sigma((d-3)^2 - 2\delta) \geq 2(\sigma - 2)\overline{g} = (\sigma - 2)(d(d-3) + 2\delta).$$

Thus,

$$(d-3)(2d - 3\sigma) \geq 4(\sigma - 1)\delta \geq 0.$$

Therefore,

$$\sigma \leq \frac{2d}{3}.$$

If $\sigma \geq d - 1$ then

$$d - 1 \leq \sigma \leq \frac{2d}{3}.$$

Hence, $d \leq 3$, which concordat the hypothesis $d \geq 4$, i.e., $d \geq 2\sigma - 1$.

In the second case,

$$eZ^2 = e((d-3)^2 - 2\delta) \geq 2\overline{g}(e - 3) = (e - 2)(d(d-3) + 2\delta)$$

and so

$$(d-3)(2d-3e) \geq 4\delta(e-1) \geq 0.$$

Hence,

$$2d \geq 3e = 3(f+\sigma) \geq 3\nu_1 + 3\sigma;$$

thus

$$3\sigma \leq 2d.$$

If $\sigma \geq d-1$ then $2d > 3\sigma \geq 3d-3$, which implies that $d \leq 1$. This contradicts the hypothesis. Hence, $\sigma \geq d-1$ cannot occur. Thus , we conclude that $d \geq 2\sigma - 1$.

If $d = 2\sigma - 1$, then $r = 0$. By $Z^2 = (\sigma-2)(\tilde{B}-4)$ and $Z^2 = (d-3)^2$, we obtain

$$\tilde{B} = 2(d-1), \tilde{B} = \frac{d+1}{2}B + 2f.$$

When one puts $B = 0$, we have $f = e = d-1$ and the type is $[\frac{d+1}{2} * (d-1); 1]$. In general, the type becomes $[\frac{d+1}{2} * (d-1); 1]$ and its similar types.

Define k to be $d - 2\sigma + 1 \geq 0$. Then $d = 2\sigma + k - 1$. Replacing d by $2\sigma + k - 1$, the formula (4) becomes

$$k(\sigma + k - 2) = 2(2\sigma - 3)\delta + \Theta_2.$$

If $r > 0$ then $k(\sigma + k - 2) \geq (k+1)(\sigma - k - 3)$ and thus,

$$\sigma \leq 2k^2 + 2k + 3.$$

Therefore, given k, we have σ such that $3 \leq \sigma \leq 2k^2 + 2k + 3$ and the equality

$$k(\sigma + k - 2) = 2(2\sigma - 3)\delta + (\sigma - 3)t_2 + 2(\sigma - 4)t_3 + 3(\sigma - 5)t_4 + \cdots .$$

holds. This equation has a finite number of non-negative solutions $\sigma, \delta, t_2, t_3, \cdots$. For example, in the cases of $k = 1, 2, 3$, we have the following solutions listed in the next tables using computer.

Table 1: types with $k = 1$

d	[$\sigma * e$, B;multiplicities]	ν_1	δ
8	$[4 * 9; 2^3]$	2	0
10	$[5 * 13, 1; 2^2]$	2	0
14	$[7 * 18, 1; 3]$	3	0

Table 2: types with $k = 2$

d	[$\sigma * e$, B;multiplicities]	ν_1	δ
9	$[4 * 13; 2^8]$	2	0
11	$[5 * 16, 1; 2^5]$	2	0
13	$[6 * 15; 2^4]$	2	0
13	$[6 * 16; 3^3]$	3	0
15	$[7 * 17; 3, 2^2]$	3	0
17	$[8 * 19; 3^2]$	3	0
19	$[9 * 25, 1; 2^3]$	2	0
19	$[9 * 21; 4, 2]$	4	0
23	$[11 * 30, 1; 3, 2]$	3	0
25	$[12 * 27; 5]$	5	0
31	$[15 * 40, 1; 4]$	4	0

Table 3: types with $k = 3$

d	$[\sigma * e\ ,\ B;$multiplicities$]$	ν_1	δ
10	$[4 * 18; 2^{15}]$	2	0
10	$[4 * 15; 2^5]$	2	1
12	$[5 * 17; 2^9]$	2	0
12	$[5 * 18, 1; 2^2]$	2	1
14	$[6 * 18; 2^7]$	2	0
14	$[6 * 19; 3^3, 2^3]$	3	0
14	$[6 * 17; 2]$	2	1
16	$[7 * 23, 1; 2^6]$	2	0
16	$[7 * 20; 3^2, 2^3]$	3	0
16	$[7 * 24, 1; 3^4]$	3	0
18	$[8 * 22; 4, 3, 2^2]$	4	0
18	$[8 * 23; 4^3]$	4	0
20	$[9 * 23; 2^5]$	2	0
20	$[9 * 28, 1; 3^3]$	3	0
20	$[9 * 28, 1; 4, 2^3]$	4	0
20	$[9 * 24; 4^2, 2]$	4	0
20	$[9 * 27, 1; 1]$	1	1

Observing these tables, we obtain the following result.

Proposition 1 *If* $P_2[D] = (d-2)(2d-5)$ *and* $\delta = g - \frac{(d-1)(d-2)}{2} \geq 0$, *then* $d \geq 4\nu_1 + 3$ *except for the following cases:*

1. $d = 8, [4 * 9; 2^3], d = 4\nu_1,$
2. $d = 10, [5 * 13, 1; 2^2], d = 4\nu_1 + 2,$
3. $d = 9, [4 * 13; 2^8], d = 4\nu_1 + 1,$
4. $d = 13, [6 * 16; 3^3], d = 4\nu_1 + 1,$
5. $d = 10, [4 * 18; 2^{15}], d = 4\nu_1 + 2,$
6. $d = 10, [4 * 15; 2^5], d = 4\nu_1 + 2,$
7. $d = 14, [6 * 19; 3^3, 2^3], d = 4\nu_1 + 2,$
8. $d = 18, [8 * 22; 4, 3, 2^2], d = 4\nu_1 + 2,$
9. $d = 18, [8 * 23; 4^3], d = 4\nu_1 + 2.$

3.2 Converse

We shall show the converse.

Proposition 2 *Suppose that nonnegative integers* $d \geq 4, \sigma, \delta, t_j (j = 2, 3, \cdots)$ *satisfy that*

$$(d - \sigma - 1)(d + 1 - 2\sigma) = 2(2\sigma - 3)\delta + \Theta_2$$

where

$$\Theta_2 = \sum_{j=2}^{\nu_1} (j-1)(\sigma - j - 1)t_j.$$

Moreover, assume that there exists a minimal pair (S, D) *obtained from a # minimal model* (Σ_B, C) *with type* $[\sigma * e, B; \nu_1, \nu_1, \cdots, \nu_r]$ *which corresponds to integers* $d, \sigma, \Delta, t_j (j = 2, 3, \cdots)$*. Then* $P_2[D] = (2d - 5)(d - 2)$.

Proof. Letting $X = \sum_{j=2}^{\nu_1} \frac{j(j-1)}{2} t_j$ and $U = \sum_{j=2}^{\nu_1} (j-1)^2 t_j$, we have $\Theta_2 = 2(\sigma - 2)X - (\sigma - 1)U$. Considering both sides of the formula (4) mod $(\sigma - 1)$, we obtain

$$\begin{aligned} (d - \sigma - 1)(d + 1 - 2\sigma) &\equiv (d-1)(d-2) && \bmod (\sigma - 1), \\ 2(2\sigma - 3)\delta + \Theta_2 &\equiv -2\delta + \Theta_2 && \bmod (\sigma - 1), \\ \Theta_2 = 2(\sigma - 2)X - (\sigma - 1)U &\equiv -2X && \bmod (\sigma - 1). \end{aligned}$$

Hence, from the formula (4), it follows that

$$(d-1)(d-2) + 2\delta + 2X \equiv 0 \bmod (\sigma - 1),$$

which implies that $\dfrac{d^2 - 3d + 2 + 2\delta + 2X}{\sigma - 1}$ is an integer. Then define $\tilde{B}_0$ to be $2 + \dfrac{d^2 - 3d + 2 + 2\delta + 2X}{\sigma - 1}$.

Now assume that there exists a minimal pair (S, D) obtained from a # minimal model (Σ_B, C) by shortest resolution of singularities, whose type is $[\sigma * e, B; \nu_1, \cdots, \nu_r]$, where $\tilde{B} = \tilde{B}_0$ and the sequence of multiplicities $\nu_2, \nu_3, \cdots$ corresponds to the sequence of $t_2, t_3, \cdots$. Indeed, when $\tilde{B}_0$ is even , one can put $B = 0, e = f = \frac{\tilde{B}_0}{2}$. Further, when $\tilde{B}_0$ is odd , σ is verified to be odd and so one can put $B = 1, e = \frac{\tilde{B}_0 + \sigma}{2}$.

By genus formula,

$$(\sigma - 1)(\tilde{B} - 2) = 2g + 2X,$$

where g is the genus of D. However, by the definition of $\tilde{B}$, we find

$$(\sigma - 1)(\tilde{B} - 2) = d^2 - 3d + 2 + 2\delta + 2X.$$

So, the genus g coincides with $d^2 - 3d + 2 + 2\delta$.

Next, we shall prove that $Z^2 = (d-3)^2 - 2\delta$.

In the previous section, we assumed $Z^2 = (d-3)^2 - 2\delta$. But here, the equation is not assumed. Define an invariant ε to be $Z^2 - ((d-3)^2 - 2\delta)$. Thus $Z^2 = \varepsilon + (d-3)^2 - 2\delta$ and then

$$(\sigma - 2)(\tilde{B} - 4) = Z^2 + U = \varepsilon + (d-3)^2 - 2\delta + U.$$

By multiplying this by $\sigma - 1$, we have

$$\begin{aligned}
&(\sigma-1)(\varepsilon + (d-3)^2) + (\sigma-1)U \\
&= (\sigma-2)((\sigma-1)(\tilde{B}-2) - 2(\sigma-1)) \\
&= (\sigma-2)(d^2 - 3d + 2 + 2\delta) + 2(\sigma-2)X - 2(\sigma-1)(\sigma-2).
\end{aligned}$$

From this , it follows that

$$\begin{aligned}
&(\sigma-1)(\varepsilon + (d-3)^2 - 2\delta) - (\sigma-2)(d^2 - 3d + 2) \\
&= (\sigma-1)\varepsilon + 2\delta(\sigma-2) + 2(\sigma-2)X - (\sigma-1)U - 2(\sigma-1)(\sigma-2).
\end{aligned}$$

Thus we obtain

$$(\sigma-1)\varepsilon + (d-\sigma-1)(d+1-2\sigma) = 2(2\sigma-3)\delta + \Theta_2. \qquad (5)$$

Recall that we assumed the equality (4). Then the formula (5) induces $(\sigma - 1)\varepsilon = 0$. Hence, $\varepsilon = 0$. Thus $Z^2 = (d-3)^2 - 2\delta$ is derived and we establish $P_2[D] = (2d-5)(d-2)$.

3.3 Examples

If $\sigma = 3$ then $\nu_1 = 1$ and the formula (4) becomes $(d-4)(d-5) = 6\delta$. Hence,

$$d \equiv 1, 2, 4, 5 \mod 6.$$

Let $[3 * e, B; 1]$ be the type. Then $Z^2 = \tilde{B} - 4 = (d-3)^2 - 2\delta$. From this it follows that

$$\tilde{B} = (d-3)^2 + 4 - \frac{(d-4)(d-5)}{3} = \frac{2d^2 - 9d + 19}{3}.$$

More precisely, when $d \equiv 1, 5 \mod 6$, it is easy to check that $\frac{2d^2-9d+19}{3}$ is even. Hence,one can put $B = 0, f = \frac{2d^2-9d+19}{6}$.

When $d \equiv 2, 4 \mod 6$, it is easy to check that $\frac{2d^2-9d+19}{3}$ is odd. Hence,one can put $B = 1, 2f + 3 = \tilde{B}$. Thus $f = \frac{2d^2-9d+10}{6}$.

Suppose that $\delta = 0$. Then $d = 4, 5$.

If $d = 4$ then $B = 1, f = 1, e = 4$. Then the type is $[3 * 4, 1; 1]$. But this contradicts the condition of $\#-$ minimality.
If $d = 5$ then $B = 0, e = f = 4$. Then the type is $[3 * 4; 1], \delta = 0$.
If $d = 7$ then $B = 0, e = 9$. Then the type is $[3 * 9; 1], \delta = 1$.
If $d = 8$ then $B = 1, e = 14$. Then the type is $[3 * 14, 1; 1], \delta = 2$.
If $d = 10$ then $B = 1, e = 23$. Then the type is $[3 * 23, 1; 1], \delta = 5$.
If $d = 11$ then $B = 0, e = 27$. Then the type is $[3 * 27; 1], \delta = 7$.

If $\sigma \geq 4$ then suppose that $r = 0$ and $\delta = 0$.
Using computer one has the following tables of types where $5 \leq d \leq 12$.

Table 4: types of pairs where $P_2[D] = (d-2)(2d-5)$ with $5 \leq d \leq 12$ and $\delta \geq 0$

d	$[\sigma * e\ ,\ B$;multiplicities$]$	ν_1	δ
5	$[3 * 4; 1]$	1	0
7	$[3 * 9; 1]$	1	1
7	$[4 * 6; 1]$	1	0
8	$[3 * 14, 1; 1]$	1	2
8	$[4 * 9; 2^3]$	2	0
9	$[4 * 13; 2^8]$	2	0
9	$[5 * 8; 1]$	1	0
10	$[3 * 23, 1; 1]$	2	5
10	$[4 * 18; 2^{15}]$	2	0
10	$[4 * 15; 2^5]$	2	1
10	$[5 * 13, 1; 2^2]$	2	0
11	$[3 * 27; 1]$	1	7
11	$[4 * 24; 2^{24}]$	2	0
11	$[4 * 21; 2^{14}]$	2	1
11	$[4 * 18; 2^4]$	2	2
11	$[5 * 16, 1; 2^5]$	2	0
11	$[6 * 10; 1]$	1	0
12	$[4 * 31; 2^{35}]$	2	0
12	$[4 * 28; 2^{25}]$	2	1
12	$[4 * 25; 2^{15}]$	2	2
12	$[4 * 22; 2^5]$	2	3
12	$[5 * 17; 2^9]$	2	0
12	$[5 * 18, 1; 2^2]$	2	1

Theorem 3 *If $4 \leq d \leq 9$, $P_2[D] = (d-2)(2d-5)$ and $g = \frac{d^2-3d+2}{2}$ then the pair (S, D) becomes a pair of $\mathbf{P^2}$ and a nonsingular plane curve or the type is $[\frac{d+1}{2} * (d-1); 1]$ or $[4 * 9; 2^3]$ or $[4 * 13; 2^8]$.*

Table 5: types of pairs where $P_2[D] = (d-2)(2d-5)$ with $13 \le d \le 15$ and $\delta = 0$

d	$[\sigma * e\ ,\ B$;multiplicities$]$	ν_1	δ
13	$[4*39; 2^{48}]$	2	0
13	$[5*21; 2^{14}]$	2	0
13	$[6*15; 2^{4}]$	2	0
13	$[6*16; 3^{3}]$	3	0
13	$[7*12; 1]$	1	0
14	$[4*48; 2^{63}]$	2	0
14	$[5*28, 1; 2^{20}]$	2	0
14	$[6*18; 2^{7}]$	2	0
14	$[6*19; 3^{3}, 2^{3}]$	3	0
14	$[7*18, 1; 3]$	3	0
15	$[4*58; 2^{80}]$	2	0
15	$[5*33, 1; 2^{27}]$	2	0
15	$[6*22; 3^{2}, 2^{8}]$	3	0
15	$[6*23; 3^{5}, 2^{4}]$	3	0
15	$[6*24; 3^{8}]$	3	0
15	$[7*17; 3, 2^{2}]$	3	0
15	$[8*14; 1]$	1	0

4 D^2 and genus

Next, suppose that $D^2 = d^2$ and $g \le \frac{(d-1)(d-2)}{2}$. So in this case $\delta_{(-)}$ is defined to be $\frac{(d-1)(d-2)}{2} - g \ge 0$. Thus $2g = d^2 - 3d + 2 - 2\delta_{(-)}$.

Assume that (S, D) is *not* birationally equivalent to $(\mathbf{P^2}, C_d)$,C_d being a nonsingular curve. Thus (S, D) is obtained from a # minimal model (Σ_B, C) of type $[\sigma * e, B; \nu_1, \nu_1, \cdots, \nu_r]$ by shortest resolution of singularities of C. Then

$$\begin{aligned} Z^2 &= K_S^2 - D^2 + 4\overline{g} \\ &= 8 - r - d^2 + 2d(d-3) - 4\delta_{(-)} \\ &= (d-3)^2 - 1 - r - 4\delta_{(-)}. \end{aligned}$$

Hence,

$$Z^2 = (d-3)^2 - 1 - r - 4\delta_{(-)}. \tag{6}$$

The genus formula implies

$$(\sigma - 1)(\tilde{B} - 2) = 2g + 2X. \tag{7}$$

Moreover,

$$\sigma\tilde{B} = D^2 + W, \tag{8}$$

where

$$W = \sum_{j=2}^{\nu_1} j^2 t_j.$$

Multiplying (7) by σ, we obtain

$$(\sigma - 1)(\sigma\tilde{B} - 2\sigma) = 2\sigma g + 2\sigma X,$$

and by (8),

$$\begin{aligned}
&(\sigma - 1)(\sigma\tilde{B} - 2\sigma) \\
&= (\sigma - 1)(D^2 + W) - 2\sigma(\sigma - 1) \\
&= 2\sigma g + 2\sigma X \\
&= (d^2 - 3d + 2)\sigma - 2\delta_{(-)}\sigma + 2\sigma X.
\end{aligned}$$

So,

$$(\sigma - 1)D^2 + (\sigma - 1)W - 2\sigma X - 2\sigma(\sigma - 1)$$

$$= (d^2 - 3d + 2)\sigma - 2\delta_{(-)}\sigma.$$

Thus, defining Θ_D to be $(\sigma - 1)W - 2\sigma X$, we have

$$\begin{aligned}
\Theta_D &= \sum_{j=2}^{\nu_1} \{(\sigma - 1)j^2 - (\sigma - 1)j(j - 1)\}t_j \\
&= \sum_{j=2}^{\nu_1} j(\sigma - j)t_j.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&- (\sigma - 1)d^2 + (d^2 - 3d + 2)\sigma + 2(\sigma - 1)(\sigma - 2) \\
&= d^2 - 3\sigma d + 2\sigma^2 \\
&= (d - \sigma)(d - 2\sigma).
\end{aligned}$$

Thus we find the following formula:

$$(d - \sigma)(d - 2\sigma) = 2\sigma\delta_{(-)} + \Theta_D. \tag{9}$$

In particular, $(d - \sigma)(d - 2\sigma) \geq 0$ implies

$$d \leq \sigma \text{ or } d \geq 2\sigma. \tag{10}$$

4.1 Estimate of d

We shall show that $d \geq 2\sigma$. Actually, by Lemma 2, we obtain either (1) $2\sigma\overline{g} \geq (\sigma-2)D^2$ or (2) $B=1$ and $eZ^2 \geq 2\overline{g}(e-3)$.

In the first case,

$$2\sigma\overline{g} = \sigma(d^2 - 3d - 2\delta_{(-)}) \geq (\sigma-2)D^2 = (\sigma-2)d^2.$$

Hence, $\overline{g} \geq 0$ and $d(2d-3\sigma) \geq 2\sigma\overline{g} \geq 0$. Thus,

$$\sigma \leq \frac{2d}{3} < d.$$

We can check $d \geq \sigma$ in the second case, too. Hence by (10), $d \geq 2\sigma$.

If $r = 0$ and $\delta_{(-)} = 0$, then $d = 2\sigma$. Since $D^2 = d^2$, it follows that $\tilde{B} = 2d$. Hence, the type becomes $[\frac{d}{2} * e, B; 1]$ such that $e = d + \frac{dB}{4}$. These types are similar to the type $[\frac{d}{2} * 2d; 1]$. Thus , if d is even, the types are $[\frac{d}{2} * 2d; 1]$ and their similar ones.

Define k to be $d - 2\sigma$. Then $d = 2\sigma + k$. We suppose that $k > 0$. Substituting $d = 2\sigma + k$, the formula (9) becomes

$$k(\sigma + k) = 2\sigma\delta_{(-)} + \Theta_D.$$

If $r > 0$ then $k(\sigma + k) \geq (k+1)(\sigma - k - 1)$. Thus,

$$\sigma \leq 2k^2 + 2k + 1.$$

For $k = 1, 2, 3$, we have the following tables.

Table 6: types in the case of $D^2 = d^2$ with $k = 1$

d	$[\sigma * e$, B;multiplicities]	ν_1	$\delta_{(-)}$
11	$[5 * 15, 1; 2]$	2	0

Table 7: types in the case of $D^2 = d^2$ with $k = 2$

d	$[\sigma * e\ ,\ B;$multiplicities$]$	ν_1	$\delta_{(-)}$
10	$[4 * 14; 2^3]$	2	0
10	$[4 * 13; 2]$	2	1
14	$[6 * 17; 2^2]$	2	0
22	$[10 * 25; 4]$	4	0
28	$[13 * 37, 1; 3]$	3	0

Table 8: types in the case of $D^2 = d^2$ with $k = 3$

d	$[\sigma * e\ ,\ B;$multiplicities$]$	ν_1	$\delta_{(-)}$
13	$[5 * 21, 1; 2^4]$	2	0
15	$[6 * 21; 3^3]$	3	0
17	$[7 * 25, 1; 2^3]$	2	0
21	$[9 * 30, 1; 3^2]$	3	0
21	$[9 * 25; 3]$	3	1
21	$[9 * 29, 1; 1]$	1	2
25	$[11 * 29; 3, 2]$	3	0
29	$[13 * 39, 1; 2]$	2	1
33	$[15 * 45, 1; 6]$	6	0
37	$[17 * 49, 1; 2^2]$	2	0
37	$[17 * 41; 5]$	5	0
53	$[25 * 69, 1; 4]$	4	0

4.2 Converse

We shall show the converse.

Proposition 3 *Suppose that nonnegative integers $d \geq 4, \sigma, \delta, t_j (j = 2, 3, \cdots)$ satisfy that*

$$(d-\sigma)(d-2\sigma) = 2\sigma\delta_{(-)} + \Theta_D,$$

where

$$\Theta_D = \sum_{j=2}^{\nu_1} j(\sigma - j)t_j.$$

Assume that there exists a minimal pair (S, D) obtained from a # minimal model (Σ_B, C) with type $[\sigma * e, B; \nu_1, \nu_1, \cdots, \nu_r]$ which corresponds to integers $d, \sigma, \Delta, t_j (j = 2, 3, \cdots)$. Then $D^2 = d^2$.

To verify this, letting $X = \sum_{j=2}^{\nu_1} \frac{j(j-1)}{2} t_j$ and $W = \sum_{j=2}^{\nu_1} j^2 t_j$, we obtain $\Theta_D = (\sigma - 1)W - 2X\sigma$ and then

$$(d-\sigma)(d-2\sigma) \equiv d^2 - 3d + 2 \mod (\sigma - 1).$$

Furthermore,

$$2\sigma\delta_{(-)} + \Theta_D \equiv 2\delta_{(-)} - 2X\sigma \mod (\sigma - 1).$$

By hypothesis,

$$\begin{aligned} 0 &= d^2 - 3d + 2 - (2\sigma\delta_{(-)} + \Theta_D) \\ &\equiv d^2 - 3d + 2 - (2\delta_{(-)} - 2X\sigma) \mod (\sigma - 1). \end{aligned}$$

Consequently, $\dfrac{d^2 - 3d + 2 - 2\delta_{(-)} + 2X}{\sigma - 1}$ is an integer. Then define

$$\tilde{B}_0 = 2 + \frac{d^2 - 3d + 2 - 2\delta_{(-)} + 2X}{\sigma - 1}.$$

By hypothesis, there exists a minimal pair (S, D) obtained from a # minimal model (Σ_B, C) with type $[\sigma * e, B; \nu_1, \nu_1, \cdots, \nu_r]$ such that $\tilde{B} = \tilde{B}_0$ and the sequence of multiplicities $\nu_2, \nu_3, \cdots$ corresponds to the sequence of $t_2, t_3, \cdots$.

By the condition , the genus g coincides with $d^2 - 3d + 2 - 2\delta_{(-)}$.

Next, we shall prove that $D^2 = d^2$. Replacing $D^2 = d^2$ by $D^2 = \varepsilon + d^2$, by the same argument as before, we obtain

$$(d-\sigma)(d-2\sigma) = 2\sigma\delta_{(-)} + \Theta_D + (\sigma - 1)\varepsilon. \tag{11}$$

Since the equality

$$(d-\sigma)(d-2\sigma) = 2\sigma\delta_{(-)} + \Theta_D$$

was assumed, it follows that $\varepsilon = 0$. Hence, $D^2 = d^2$.

4.3 Examples

If $\sigma = 3$ then $\nu_1 = 1$ and the formula becomes $(d-3)(d-6) = 6\delta_{(-)}$. Hence,

$$d \equiv 0 \mod 3.$$

By $[3 * e, B; 1]$ we denote the type. Then $D^2 = 3\tilde{B}$ and therefore, $\tilde{B} = \frac{d^2}{3}$.

When $d = 3\mu$, we have $\tilde{B} = 3\mu^2$ and $\delta = \frac{3(\mu-1)(\mu-2)}{2}$. Hence, if d is even, then put $B = 0$ and thus $f = \frac{3\mu^2}{2}$. The type is $[3 * \frac{3\mu^2}{2}; 1]$ (or its similar ones).

If d is odd, then put $B = 1$ and thus $f = \frac{3\mu^2-3}{2}$. The type is $[3 * \frac{3\mu^2+3}{2}, 1; 1]$.

Suppose that $\delta_{(-)} = 0$. Then $d = 6$ and so by putting $B = 0$, we get $e = 6$ and the type becomes $[3 * 6; 1]$.

In general, if $d = 9$, then $B = 1, e = 15, \delta_{(-)} = 3$ and so the type is $[3 * 15, 1; 1]$.

If $d = 12$, then $B = 0, e = 24, \delta_{(-)} = 9$ and so the type is $[3 * 24; 1]$.

Suppose that $r = 0$ and $\delta_{(-)} = 0$. Then by the formula, $d = 2\sigma$. In particular, d is even. Hence,$\sigma = \frac{d}{2}$. By $D^2 = \sigma\tilde{B} = d^2$, we obtain

$$\tilde{B} = 2d, \quad \tilde{B} = \frac{d}{2}B + 2f.$$

When $B = 0$, we have $f = e = d$ and the type is $[\frac{d}{2} * d; 1]$. In general, the type becomes $[\frac{d}{2} * d; 1]$ and its similar ones.

Using computer, one has the following tables of types where $5 \leq d \leq 12$.

Observing these formulas, we obtain the next proposition.

Theorem 4 *Suppose that* $D^2 = d^2$ *and* $g = \frac{(d-1)(d-2)}{2}$.

Then whenever $d = 4, 5, 7, 9$*, the pair is birationally equivalent to* $(\mathbf{P^2}, C_d)$*,* C_d *being a nonsingular curve.*

5 Z^2 and D^2

Suppose that $Z^2 = (d-3)^2$ and $D^2 \geq d^2$ for some $d \geq 4$. Then Δ is defined to be $D^2 - d^2$, which is nonnegative. $g - \frac{(d-1)(d-2)}{2}$ is denoted by δ, which will be proved to be positive.

Assume that (S, D) is *not* birationally equivalent to $(\mathbf{P^2}, C_d)$,C_d being a nonsingular curve. Thus (S, D) is obtained from a # minimal model (Σ_B, C) of type $[\sigma * e, B; \nu_1, \nu_1, \cdots, \nu_r]$ by shortest resolution of singularities of C. Then from

$$Z^2 = K_S^2 - D^2 + 4\overline{g},$$

it follows that

$$(d-3)^2 = Z^2 = 8 - r - (d^2 + \Delta) + 2d(d-3) + 4\delta.$$

Table 9: types in the case of $D^2 = d^2$ with $4 \leq d \leq 13$

d	$[\sigma * e\ ,\ B$;multiplicities$]$	ν_1	$\delta_{(-)}$
6	$[3 * 6; 1]$	1	0
8	$[4 * 8; 1]$	1	0
9	$[3 * 15, 1; 1]$	1	3
10	$[4 * 14; 2^3]$	2	0
10	$[4 * 21; 2]$	2	1
10	$[5 * 10; 1]$	1	0
11	$[5 * 15, 1; 2]$	2	0
12	$[3 * 24; 1]$	1	9
12	$[4 * 22; 2^8]$	2	0
12	$[4 * 22; 2^8]$	2	0
12	$[4 * 29; 2^6]$	2	1
12	$[4 * 36; 2^4]$	2	2
12	$[4 * 43; 2^2]$	2	3
12	$[4 * 50; 1]$	1	4
12	$[6 * 12; 1]$	1	0

Hence,

$$4\delta = 1 + r + \Delta. \tag{12}$$

Multiplying (3) by σ, we obtain

$$\begin{aligned}\sigma Z^2 + \sigma U &= (\sigma - 2)(\sigma \tilde{B} - 4\sigma) \\ &= (\sigma - 2)(D^2 + W) - 4(\sigma - 2)\sigma \\ &= (\sigma - 2)d^2 + (\sigma - 2)\Delta + (\sigma - 2)W - 4(\sigma - 2)\sigma.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma Z^2 + \sigma U &= \sigma(d - 3)^2 + \sigma U \\ &= (\sigma - 2)d^2 + (\sigma - 2)\Delta + (\sigma - 2)W - 4(\sigma - 2)\sigma,\end{aligned}$$

and so

$$\sigma(d - 3)^2 - (\sigma - 2)d^2 + 4(\sigma - 2)\sigma = (\sigma - 2)\Delta + (\sigma - 2)W - \sigma U.$$

Defining

$$\Theta_{DZ} = (\sigma - 2)W - \sigma U,$$

we have

Table 10: types in the case of $D^2 = d^2$ with $13 \leq d \leq 18$ where $r > 0, \delta_{(-)} = 0$

d	$[\sigma * e\ ,\ B;\ \text{Type}]$	ν_1	$\delta_{(-)}$
13	$[5 * 21, 1; 2^4]$	2	0
14	$[4 * 32; 2^{15}]$	2	0
14	$[5 * 22; 2^6]$	2	0
14	$[6 * 17; 2^2]$	2	0
15	$[6 * 21; 3^3]$	3	0
16	$[4 * 44; 2^{24}]$	2	0
16	$[5 * 30; 2^{11}]$	2	0
16	$[6 * 23; 2^5]$	2	0
17	$[5 * 37, 1; 2^{14}]$	2	0
17	$[7 * 25, 1; 2^3]$	2	0
18	$[4 * 58; 2^{35}]$	2	0
18	$[6 * 30; 2^9]$	2	0
18	$[6 * 33; 3^8]$	3	0
18	$[7 * 25; 3^2, 2^2]$	3	0
19	$[5 * 47, 1; 2^{21}]$	2	0
19	$[6 * 35; 3^3, 2^8]$	3	0
19	$[7 * 31, 1; 2^6]$	2	0
19	$[7 * 29; 3^5]$	3	0
20	$[4 * 74; 2^{48}]$	2	0
20	$[5 * 50; 2^{25}]$	2	0
20	$[6 * 38; 2^{14}]$	2	0
20	$[6 * 41; 3^8, 2^5]$	3	0
20	$[7 * 32; 3^4, 2^3]$	3	0
20	$[8 * 26; 2^4]$	2	0
20	$[8 * 28; 4^3]$	4	0
21	$[6 * 45; 3^7, 2^9]$	3	0
21	$[6 * 48; 3^{15}]$	3	0
21	$[7 * 39, 1; 3^4, 2^5]$	3	0
21	$[9 * 30, 1; 3^2]$	3	0

$$\Theta_{DZ} = \sum_{j=2}^{\nu_1}(-2j^2 + 2\sigma j - \sigma)t_j \geq \sum_{j=2}^{\nu_1} 2j(j-1)t_j.$$

Thus, noting that

$$\sigma(d-3)^2 - (\sigma-2)d^2 + 4(\sigma-2)\sigma = 2d^2 - 6\sigma d + (4\sigma+1)\sigma,$$

we find the next formula:

$$2d^2 - 6\sigma d + (4\sigma+1)\sigma = (\sigma-2)\Delta + \Theta_{DZ}, \tag{13}$$

where $\Theta_{DZ} = \sum_{j=2}^{\nu_1}(-2j^2 + 2\sigma j - \sigma)t_j$.

Claim: If $\Theta_{DZ} = 0$ then $\Delta \geq 3$.
Actually, $\Theta_{DZ} = 0$ implies $r = 0$. But, from $4\delta = 1 + r + \Delta = 1 + \Delta$, it follows that $\Delta \geq 3$.

By the Claim, $(\sigma-2)\Delta + \Theta_{DZ} > 0$ and so

$$2d^2 - 6\sigma d + (4\sigma+1)\sigma \geq 1.$$

Moreover,

$$2d^2 - 6\sigma d + (4\sigma+1)\sigma - \frac{1}{2} = \frac{(2d-4\sigma+1)(2d-2\sigma-1)}{2} \geq \frac{1}{2}.$$

Hence,

$$(2d-4\sigma+1)(2d-2\sigma-1) > 0. \tag{14}$$

Therefore, we have either $2d \leq 2\sigma+1$ or $2d \geq 4\sigma-1$ and so we obtain either 1) $\sigma \geq d$ or 2) $d \geq 2\sigma$.

5.1 Estimate of d

We shall show that $d \geq 2\sigma$.
Actually, by Lemma 2 , we have either (1) $\sigma Z^2 \geq 2(\sigma-2)\overline{g}$ or (2) $B = 1$ and $eZ^2 \geq 2(e-3)\overline{g}$.

In the first case ,

$$\sigma(d-3)^2 = \sigma Z^2 \geq 2(\sigma-2)\overline{g} = (\sigma-2)(d(d-3)+2\delta) \geq (\sigma-2)d(d-3).$$

Therefore, $2d \geq 3\sigma$, and so $\sigma \leq \frac{2d}{3}$; hence by (14), we obtain $d \geq 2\sigma$.

In the second case , it follows that

$$e(d-3)^2 = eZ^2 \geq 2(e-3)\overline{g} = (e-3)d(d-3).$$

Hence, $e(d-3) \geq (e-3)d$, which implies that $d \geq e = f + \sigma > \sigma$. Therefore,

$$\sigma \leq d-1.$$

Hence, by (14), we obtain

$$2d - 4\sigma + 1 > 0; \quad d \geq 2\sigma.$$

Suppose that $d = 2\sigma$. Then the formula (12) turns out to be

$$2d^2 - 6\sigma d + (4\sigma+1)\sigma = \sigma = (\sigma-2)\Delta + \Theta_{DZ}.$$

Since

$$\sigma = (\sigma-2)\Delta + \Theta_{DZ} \geq \Theta_{DZ} \geq (2\nu_1 - 1)\sigma - 2\nu_1^2,$$

it follows that

$$\nu_1^2 \geq 2(\nu_1 - 1)\nu_1.$$

Hence, $2 \geq \nu_1$.

Assume that $\nu_1 = 2$. Then $\sigma = 4, d = 8; \Theta_{DZ} = 4, t_2 = 1, \Delta = 0$. Hence,

$$D^2 = \sigma\tilde{B} - 4 = d^2 = 64.$$

Thus, $\tilde{B} = 17$ and $17 = \tilde{B} = 2f + 4B$, which is a contradiction.

Assume that $\nu_1 = 1$. Then $r = 0, 4\delta = 1 + \Delta \geq 4$ and

$$\sigma = (\sigma-2)\Delta \geq 3(\sigma-2).$$

Hence, $\sigma = 3, d = 6, e = 8, \Delta = 3$, which imply that the type is $[3*8,1;1]$.

Define k to be $d - 2\sigma$. Replacing d by $2\sigma + k$, the formula (13) turns out to be

$$2k^2 + (2k+1)\sigma = (\sigma-2)\Delta + \Theta_{DZ}. \quad (15)$$

Since

$$2k^2 + (2k+1)\sigma \geq (-2j^2 + 2\sigma j - \sigma), j = k+2$$

it follows that $\sigma \leq 2(k^2 + 2k + 2)$. Thus , we obtain the following tables using computer.

By observing these tables, we obtain the following result.

Proposition 4 *If $D^2 = d^2$ and $Z^2 = (d-3)^2$ and (S, D) is not birationally equivalent to pairs of the projective plane and non-singular curves, then*

$$d \geq 4\nu_1 + 3$$

except for the type $[6*25,1;3^5]$.

Table 11: types in the case of $D^2 = d^2$ and $Z^2 = (d-3)^2$ with $k = 1, 2, 3$

d	$[\sigma * e\ ,\ B$;multiplicities$]$	ν_1	Δ
21	$[10 * 27, 1; 3]$	3	0
10	$[4 * 16; 2^7]$	2	0
16	$[7 * 23, 1; 3, 2^2]$	3	0
18	$[8 * 21; 2^3]$	2	0
42	$[20 * 54, 1; 4]$	4	0
15	$[6 * 25, 1; 3^5]$	3	0
21	$[9 * 26; 3^3]$	3	0
21	$[9 * 31, 1; 4^2, 2]$	4	0
25	$[11 * 35, 1; 4, 2^2]$	4	0
29	$[13 * 33; 3, 2^2]$	3	0
39	$[18 * 53, 1; 9]$	9	0
45	$[21 * 59, 1; 2^3]$	2	0
71	$[34 * 91, 1; 5]$	5	0

5.2 Converse

By the same argument as in the previous section, we can show the converse.

Proposition 5 *Suppose that nonnegative integers $d, \sigma, \Delta, t_j (j = 2, 3, \cdots)$ satisfy that*

$$2d^2 - 6\sigma d + (4\sigma + 1)\sigma = (\sigma - 2)\Delta + \Theta_{DZ} \tag{16}$$

and that $\Delta + 1 + r$ is even.

Assume that there exists a minimal pair (S, D) obtained from a # minimal model (Σ_B, C) with type $[\sigma * e, B; \nu_1, \nu_1, \cdots, \nu_r]$ which corresponds to integers $d, \sigma, \Delta, t_j (j = 2, 3, \cdots)$. Then $Z^2 = (d-3)^2$.

Proof. By (14),

$$2d^2 - 6\sigma d + (4\sigma + 1)\sigma \equiv 2d^2 + \sigma \mod 2\sigma.$$

Hence,

$$2d^2 + \sigma \equiv (\sigma - 2)\Delta + (\sigma - 2)W - \sigma U \mod 2\sigma.$$

Thus

$$2(d^2 + \Delta + W) \equiv \sigma(\Delta + W - U - 1) \mod 2\sigma.$$

By the way,

$$W - U = \sum_{j=2}^{\nu_1} \{j^2 - (j-1)^2\} t_j$$

and

$$W - U - r = \sum_{j=2}^{\nu_1} \{j^2 - (j-1)^2 - 1\} t_j \equiv 0 \mod 2.$$

Therefore,

$$\begin{aligned}\sigma(\Delta + W - U - 1) &= \sigma(\Delta + W - U - r) + \sigma(r - 1) \\ &\equiv \sigma(\Delta + r - 1) \bmod 2\sigma.\end{aligned}$$

However, since $\Delta + 1 + r$ is even, it follows that

$$\sigma(\Delta + r - 1) \equiv 0 \bmod 2\sigma.$$

So,

$$\sigma(\Delta + W - U - 1) \equiv 0 \bmod 2\sigma. \tag{17}$$

Therefore,

$$2(d^2 + \Delta + W) \equiv 0 \mod 2\sigma,$$

which implies that $\dfrac{d^2 + \Delta + W}{\sigma}$ is an integer, which we denote by $\tilde{B}_0$. Thus,

$$\sigma \tilde{B}_0 = d^2 + \Delta + W. \tag{18}$$

As in the previous sections, assume that there exists a minimal pair (S, D) obtained from a # minimal model (Σ_B, C) with type $[\sigma * e, B; \nu_1, \nu_1, \cdots, \nu_r]$ of which $\tilde{B}$ equals $\tilde{B}_0$ and the sequence of multiplicities $\nu_2, \nu_3, \cdots$ corresponds to the sequence of $t_2, t_3, \cdots$. Then

$$\sigma \tilde{B} = D^2 + W, \sigma \tilde{B} = \sigma \tilde{B}_0 = d^2 + \Delta + W.$$

Defining ε to be $Z^2 - (d-3)^2$, we have

$$(\sigma - 2)(\tilde{B} - 4) = (d-3)^2 + \varepsilon + U.$$

Multiplying the above formula by σ, we obtain

$$(\sigma - 2)(\sigma \tilde{B} - 4\sigma) = \sigma(d-3)^2 + \sigma\varepsilon + \sigma U$$

and

$$\begin{aligned}(\sigma - 2)(\sigma \tilde{B} - 4\sigma) &= (\sigma - 2)(D^2 + W) - 4\sigma(\sigma - 2) \\ &= (\sigma - 2)(d^2 + \Delta + W) - 4\sigma(\sigma - 2).\end{aligned}$$

Therefore,

$$(\sigma-2)(d^2+\Delta+W)-4\sigma(\sigma-2)=\sigma((d-3)^2+\varepsilon)+\sigma U.$$

Hence,

$$\sigma\varepsilon=2d^2-6\sigma d+(4\sigma+1)\sigma-(\sigma-2)\Delta-\Theta_{DZ}.$$

However, the formula (16) implies that the right hand side vanishes. Hence,

$$\sigma\varepsilon=0; \quad \varepsilon=0.$$

Therefore, $Z^2=(d-3)^2$ has been established.

5.3 Numerical examples

Table 12: types in the case of $D^2 = d^2$ and $Z^2 = (d-3)^2$ with $4 \le d \le 21$

d	[$\sigma * e$, B;multiplicities]	Δ	ν_1
10	$[4 * 16; 2^7]$	0	2
14	$[4 * 40; 2^{31}]$	0	2
14	$[5 * 24; 2^{11}]$	0	2
15	$[5 * 31, 1; 2^{15}]$	0	2
16	$[7 * 23, 1; 3, 2^2]$	0	3
17	$[6 * 29; 3^3, 2^8]$	0	3
18	$[4 * 76; 2^{71}]$	0	2
18	$[6 * 32; 2^{15}]$	0	2
18	$[7 * 29, 1; 3, 2^6]$	0	3
18	$[8 * 21; 2^3]$	0	2
20	$[6 * 41; 2^{23}]$	0	2
20	$[7 * 37, 1; 3^5, 2^6]$	0	3
21	$[5 * 67, 1; 2^{51}]$	0	2
21	$[6 * 47; 3^3, 2^{24}]$	0	3
21	$[6 * 54; 3^{23}]$	0	3
21	$[7 * 40, 1; 3^2, 2^{13}]$	0	3
21	$[8 * 31; 4, 3^3, 2^3]$	0	4
21	$[9 * 26; 3^3]$	0	3
21	$[9 * 31, 1; 4^2, 2]$	0	4

Observing these tables, we get the next result.

Theorem 5 (H.Yanaba) *Suppose that $Z^2 = (d-3)^2$ and $D^2 = d^2$.*

If $d = 4, 5, 7, 8, 9, 11, 12, 13, 19$, then (S, D) is birationally equivalent to a pair of $\mathbf{P}^2$ and a nonsingular curve.

6 $P_{3,1}[D]$ and genus

Suppose that $P_{3,1}[D] = \frac{(d-7)(d-8)}{2}$ for $d \geq 7$, and $\delta = g - \frac{(d-1)(d-2)}{2} \geq 0$, g being the genus of D. Then assume that a minimal pair (S, D) is *not* birationally equivalent to $(\mathbf{P^2}, C_d)$,C_d being a nonsingular curve. Then (S, D) is obtained from a # minimal model (Σ_B, C) of type $[\sigma * e, B; \nu_1, \nu_1, \cdots, \nu_r]$ by shortest resolution of singularities of C. By the same argument as before,

$$(\sigma - 1)(\tilde{B} - 2) = 2g + 2X. \tag{19}$$

Moreover, assuming $\sigma \geq 6$, we have $2P_{3,1}[D] - 2 = (3Z - 2D)(2Z - D)$. Thus

$$(\sigma - 5)(\tilde{B} - 10) - 2 = (3Z - 2D)(2Z - D) + 2Y. \tag{20}$$

Here, $Y = \sum_{j=2}^{\nu_1} \frac{(j-2)(j-3)}{2} t_j$. Then

$$(\sigma - 5)(\tilde{B} - 10) = (d - 7)(d - 8) + 2Y. \tag{21}$$

Multiplying (21) by $\sigma - 1$, we obtain

$$\begin{aligned}&(\sigma - 5)(\sigma - 1)(\tilde{B} - 2) - 8(\sigma - 1)(\sigma - 5)\\ &= (\sigma - 1)(d - 7)(d - 8) + 2(\sigma - 1)Y.\end{aligned}$$

From hypothesis, it follows that

$$(\sigma - 5)(\sigma - 1)(\tilde{B} - 2) = (\sigma - 5)(d^2 - 3d + 2) + 2\delta(\sigma - 5) + 2X(\sigma - 5).$$

Hence, defining Θ_{31} to be $(\sigma - 5)X - (\sigma - 1)Y$, we have

$$\begin{aligned}\Theta_{31} &= \sum_{j=2}^{\nu_1} \{\sigma(2j - 3) - 2j^2 + 3\} t_j \geq (\sigma - 5)t_2\\ &+ \sum_{j=3}^{\nu_1} \{2j(j - 3) + 3\} t_j.\end{aligned}$$

Note that $\Theta_{31} = 0$ implies $r = 0$.

Moreover,

$$\begin{aligned}&(\sigma - 1)(d - 7)(d - 8) - (\sigma - 5)(d^2 - 3d + 2) + 8(\sigma - 1)(\sigma - 5)\\ &= 2d^2 - 6\sigma d + 4\sigma^2 + 3\sigma - 3.\end{aligned}$$

Consequently,

$$2d^2 - 6\sigma d + 4\sigma^2 + 3\sigma - 3 = \delta(\sigma - 5) + \Theta_{31}. \tag{22}$$

So,

$$2d^2 - 6\sigma d + 4\sigma^2 + 3\sigma - 3 \geq 0.$$

However,

$$\begin{aligned}
&2d^2 - 6\sigma d + 4\sigma^2 + 3\sigma - 3 \\
&= 2d^2 - 6\sigma d + 4\sigma^2 + 3\sigma - \frac{9}{2} - \frac{1}{2} \\
&= \frac{(2d - 4\sigma + 3)(2d - 2\sigma - 3)}{2} - \frac{1}{2} \geq 0.
\end{aligned}$$

Hence, $(2d - 4\sigma + 3)(2d - 2\sigma - 3) > 0$.

Therefore, we have either $\sigma > \frac{2d-3}{2}$ or $\sigma < \frac{2d+3}{4}$. From $\sigma > \frac{2d-3}{2}$, it follows that $d \leq \sigma + 1$. Similarly, $\sigma < \frac{2d+3}{4}$ implies $d \geq 2\sigma - 1$.

6.1 Estimate of d

We shall verify that if $d \leq \sigma + 1$ then $d = \sigma + 1$ and the type is either 1) $[6 * 8, 1; 2^r], r \leq 5, d = 7$ or 2) $[7 * 9, 1; 2^r], r \leq 6, d = 8$. Otherwise, $d \geq 2\sigma$.

Actually, assuming $d \leq \sigma + 1$, by Lemma 1 we have either (1) $|\sigma Z - (\sigma - 2)D| \neq \emptyset$ or (2) $B = 1, 2f < \sigma$ and $|eZ - (e - 3)D| \neq \emptyset$.

In the first case, since $\sigma \geq 4$, it follows that $2Z - D$ is nef. Hence,

$$(\sigma Z - (\sigma - 2)D) \cdot (2Z - D) \geq 0$$

and

$$2\sigma Z^2 + (\sigma - 2)D^2 + 2(4 - 3\sigma)\overline{g} \geq 0. \tag{23}$$

By hypothesis,

$$6Z^2 + 2D^2 - 14\overline{g} = (d - 7)(d - 8). \tag{24}$$

Eliminating D^2 from these two formulas, we obtain

$$(6 - \sigma)Z^2 + \frac{(\sigma - 2)(d - 7)(d - 8)}{2} \geq (6 - \sigma)\overline{g}.$$

Hence,

$$\frac{(\sigma - 2)(d - 7)(d - 8)}{2} \geq (\sigma - 6)(Z^2 - \overline{g}).$$

But by Lemma 2,

$$\sigma Z^2 \geq 2(\sigma - 2)\overline{g}$$

and so

$$Z^2 \geq 2(1 - \frac{2}{\sigma})\overline{g}.$$

Therefore,

$$(\sigma - 6)(Z^2 - \overline{g}) \geq (\sigma - 6)(1 - \frac{4}{\sigma})\overline{g}.$$

Hence,

$$\sigma(\sigma - 2)(d - 7)(d - 8) \geq (\sigma - 4)(\sigma - 6)d(d - 3). \tag{25}$$

Defining a quadratic equation $F(x)$ by

$$\sigma(\sigma - 2)(x - 7)(x - 8) - (\sigma - 4)(\sigma - 6)x(x - 3),$$

we shall verify that if $F(d) \geq 0$ then $d \geq \sigma + 1$.

This follows from observing Figure 1 which is the figure of curves defined by $x(x-2)(y-7)(y-8) = (x-4)(x-6)y(y-3), x = 6, y = 6, y = x + 1$.

If $d = \sigma + 1$ then the formula (23) induces

$$(d - 1)(d - 3)(d - 7)(d - 8) \geq (d - 5)(d - 7)d(d - 3),$$

which implies either $d = 7$ or

$$(d - 1)(d - 8) \geq d(d - 5).$$

Then $-9d + 8 \geq -5d$; $2 \geq d$. But this is impossible.

If $d = 7$ then $\sigma = 6$ and by (21) we have $\tilde{B} = 10$. Hence, the type becomes $[6 * 8, 1; 2^r]$.

In the second case, since $|eZ - (e - 3)D| \neq \emptyset$ and $2Z - D$ is nef for $\sigma \geq 4$, it follows that

$$(eZ - (e - 3)D) \cdot (2Z - D) \geq 0.$$

Therefore,

$$2eZ^2 + (e - 3)D^2 + 2(6 - 3e)\overline{g} \geq 0. \tag{26}$$

Recalling (24), we obtain

$$(9 - e)Z^2 + \frac{(d - 7)(d - 8)(e - 3)}{2} \geq (9 - e)\overline{g}.$$

Hence,

$$\frac{(d - 7)(d - 8)(e - 3)}{2} \geq (e - 9)(Z^2 - \overline{g}). \tag{27}$$

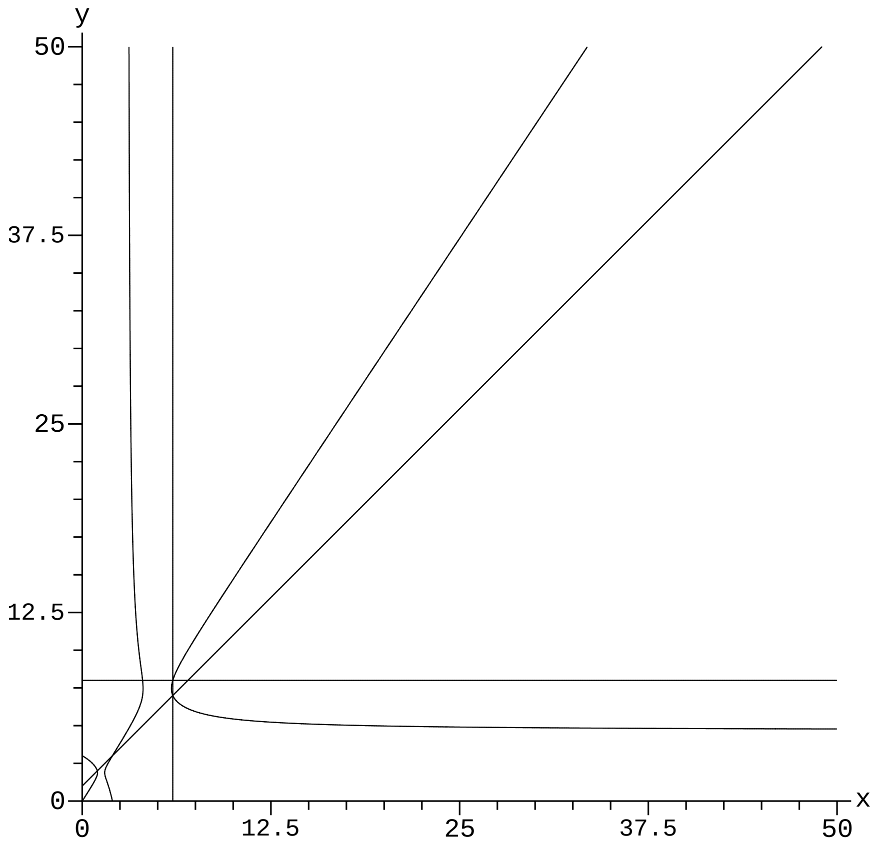

Figure 1: $x(x-2)(y-7)(y-8)=(x-4)(x-6)y(y-3), x=6, y=6, y=x+1$

But by Lemma2,

$$Z^2 - \overline{g} \geq \frac{e-6}{e}\overline{g} \geq \frac{(e-6)d(d-3)}{2e}.$$

Combining this with (26), we obtain

$$e(e-3)(d-7)(d-8) \geq (e-6)(e-9)d(d-3). \tag{28}$$

Noting that $d \geq 8$ and $e \geq 9$, we have the next figure of curves.

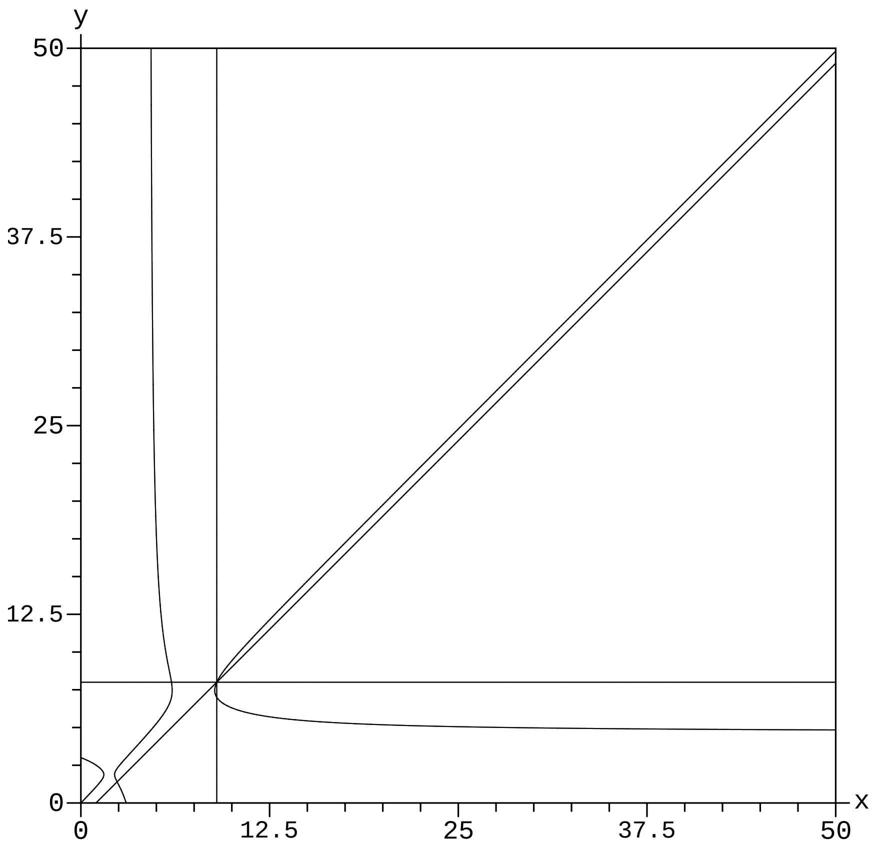

Figure 2: $x(x-3)(y-7)(y-8) = (x-6)(x-9)y(y-3), x = 9, y = 8, y = x-1$

Observing Figure 2, we get $d \geq e-1$. Since $e \geq \sigma + \nu_1$, we get

$$d \geq e - 1 = f + \sigma - 1 \geq f + \sigma - 1 \geq \sigma + \nu_1 - 1.$$

Suppose that $d = \sigma + 1$. Then $d = e - 1$ and by (27) ,we obtain

$$e(e-3)(e-8)(e-9) \geq (e-6)(e-9)(e-1)(e-4).$$

Hence, either $e = 9$ or

$$e(e-3)(e-8) \geq (e-6)(e-1)(e-4).$$

This induces $24 \geq 10e$; hence, $2 \geq e$, which is a contradiction. Thus $e = 9$ and so $d = 8, \sigma = 7$ and the type is $[7 * 9, 1; 2^r]$, where $r \leq 6, d = 7$.

Given d and σ, one can enumerate $\delta, t_2, t_3, \cdots$ satisfying the following formula:

$$(\sigma - 5)\delta + \Theta_{31} = (\sigma - 5)(\delta + t_2 + 3t_3) + (5\sigma - 29)t_4 + \cdots .$$

Since $\delta + t_2 + 3t_3$ is invariant, if d and σ are given,then in the following table $t_3 = 0, \delta = 0$ is assumed. For example,if the type $[8* \ 17;2^7]$ is given, other types such as $[8* \ 17;3^{t_3}, 2^{t_2}]$ with $7 = \delta + t_2 + 3t_3$ exist.

6.2 Numerical examples

Table 13: types where $2P_{3,1}[D] = (d-7)(d-8), 2g = (d-1)(d-2)$ with $7 \leq d \leq 19$ and $t_3 = 0, \delta = 0$

d	$[\sigma * e\ ,\ B$;multiplicities$]$	ν_1	δ
7	$[6*8; 2^5]$	2	0
8	$[7*9; 2^6]$	2	0
11	$[6*11; 2^5]$	2	0
12	$[6*15; 2^{15}]$	2	0
13	$[6*20; 2^{29}]$	2	0
13	$[7*16, 1; 2^3]$	2	0
14	$[6*26; 2^{47}]$	2	0
14	$[7*19, 1; 2^9]$	2	0
15	$[6*33; 2^{69}]$	2	0
15	$[7*19; 2^{17}]$	2	0
16	$[6*41; 2^{95}]$	2	0
16	$[7*23; 2^{27}]$	2	0
16	$[8*17; 2^7]$	2	0
17	$[6*50; 2^{125}]$	2	0
17	$[7*31, 1; 2^{39}]$	2	0
17	$[8*20; 2^{13}]$	2	0
17	$[8*21; 4^3, 2^2]$	4	0
18	$[6*60; 2^{159}]$	2	0
18	$[7*36, 1; 2^{53}]$	2	0
18	$[8*24; 4^2, 2^{13}]$	4	0
18	$[8*25; 4^5, 2^2]$	4	0
18	$[9*19; 4, 2^2]$	4	0
19	$[6*71; 2^{197}]$	2	0
19	$[7*38; 2^{69}]$	2	0
19	$[8*27; 2^{29}]$	2	0
19	$[8*28; 4^3, 2^{18}]$	4	0
19	$[8*29; 4^6, 2^7]$	4	0
19	$[9*26, 1; 2^{11}]$	2	0
19	$[9*22; 4^2, 2^3]$	4	0

Table 14: types where $2P_{3,1}[D] = (d-7)(d-8), 2g = (d-1)(d-2)$ with $20 \leq d \leq 21$, and $t_3 = 0, \delta = 0$

d	$[\sigma * e\ ,\ B$;multiplicities]	ν_1	δ
20	$[6*83; 2^{239}]$	2	0
20	$[7*44; 2^{87}]$	2	0
20	$[8*31; 2^{39}]$	2	0
20	$[8*32; 4^3, 2^{28}]$	4	0
20	$[8*33; 4^6, 2^{17}]$	4	0
20	$[8*34; 4^9, 2^6]$	4	0
20	$[9*29, 1; 2^{17}]$	2	0
20	$[9*25; 4^2, 2^9]$	4	0
20	$[9*30, 1; 4^4, 2]$	4	0
21	$[6*96; 2^{285}]$	2	0
21	$[7*54, 1; 2^{107}]$	2	0
21	$[8*36; 4^2, 2^{43}]$	4	0
21	$[8*37; 4^5, 2^{32}]$	4	0
21	$[8*38; 4^8, 2^{21}]$	4	0
21	$[8*39; 4^{11}, 2^{10}]$	4	0
21	$[9*28; 4, 2^{20}]$	4	0
21	$[9*33, 1; 4^3, 2^{12}]$	4	0
21	$[9*29; 4^5, 2^4]$	4	0
21	$[10*24; 5, 4, 2]$	5	0

7 $P_{2,1}[D]$ and $P_{3,1}[D]$

Suppose that a minimal pair (S,D) satisfies that $P_{2,1}[D] \geq \frac{(d-4)(d-5)}{2}$ and $P_{3,1}[D] = \frac{(d-7)(d-8)}{2}$ for $d > 6$ that is *not* birationally equivalent to $(\mathbf{P^2}, C_d)$,C_d being a nonsingular curve. Then (S,D) is obtained from a # minimal model (Σ_B, C) of type $[\sigma * e, B; \nu_1, \nu_1, \cdots, \nu_r]$ by shortest resolution of singularities of C. Then defining Δ_{21} to be $P_{2,1}[D] - \frac{(d-4)(d-5)}{2} \geq 0$,

$$(\sigma - 3)(\tilde{B} - 6) = (d-4)(d-5) + 2\Delta_{21} + 2V. \tag{29}$$

Here, $V = \sum_{j=2}^{\nu_1} \frac{(j-2)(j-1)}{2} t_j$. Moreover,

$$(\sigma - 5)(\tilde{B} - 10) = (d-7)(d-8) + 2Y. \tag{30}$$

Here, $Y = \sum_{j=2}^{\nu_1} \frac{(j-2)(j-3)}{2} t_j$.

Then multiplying (27) by $\sigma - 3$, we obtain

$$(\sigma - 3)(\sigma - 5)(\tilde{B} - 10) = (\sigma - 3)(d-7)(d-8) + 2(\sigma - 3)Y. \tag{31}$$

By (26),

$$\begin{aligned}&(\sigma - 3)(\sigma - 5)(\tilde{B} - 10)\\ &= (\sigma - 3)(\sigma - 5)(\tilde{B} - 6) - 4(\sigma - 3)(\sigma - 5)\\ &= (\sigma - 5)((d-4)(d-5) + 2\Delta_{21} + 2V) + (\sigma - 5)\Delta_{21} - 4(\sigma - 3)(\sigma - 5).\end{aligned}$$

Hence,

$$\begin{aligned}&(\sigma - 3)(d-7)(d-8) + 2(\sigma - 5)Y\\ &= (\sigma - 5)((d-4)(d-5) + 2V) + (\sigma - 5)\Delta_{21} - 4(\sigma - 3)(\sigma - 5).\end{aligned}$$

Therefore, defining Θ_{32} to be $(\sigma - 3)V - (\sigma - 5)Y$, we obtain

$$(d - \sigma - 2)(d + 2 - 2\sigma) = (\sigma - 5)\Delta_{21} + \Theta_{32}. \tag{32}$$

Here, $\Theta_{32} = \sum_{j=3}^{\nu_1} (j-2)(\sigma - j - 2)t_j = (\sigma - 5)t_3 + 2(\sigma - 6)t_4 + \cdots$.

Since $\Theta_{32} \geq 0$, it follows that

$$(d - \sigma - 2)(d + 2 - 2\sigma) \geq 0.$$

Thus either $d \leq \sigma + 2$ or $d \geq 2\sigma - 2$.

Note that if $(\sigma - 5)\Delta_{21} + \Theta_{32} = 0$ and $\sigma \geq 6$ then $\Delta_{21} = 0$ and $\nu_1 \leq 2$. Moreover, in this case, we have two cases: $d = \sigma + 2$ or $d = 2\sigma - 2$.

If $\sigma = d-2$, then the type becomes $[\sigma * (\sigma+2), 1; 2^r]$.

If $\sigma = \frac{d+2}{2}$, then $d = 2\sigma - 2$ and from the formula

$$(\sigma-3)(\tilde{B}-6) = (d-4)(d-5) + 2V = 2(\sigma-3)(2\sigma-7)$$

it follows that $\tilde{B} = 4\sigma - 8$.

When $B = 0$, we have $f = 2\sigma - 4$. The type becomes $[\sigma * 2(\sigma-2); 2^r]$.

When $B = 1$, we have $2f = 3\sigma - 8$. Then σ is even and the type becomes $[\sigma * \frac{5\sigma-8}{2}, 1; 2^r]$.

7.1 Estimate of d

We shall verify that if $\sigma \geq d-2$, then $B = 1, f = 2, d = \sigma + 2$ and the type is $[(d-2) * d, 1; 2^r]$.

Actually, $P_{2,1}[D] = \frac{(d-4)(d-5)}{2}$ and $P_{3,1}[D] = \frac{(d-7)(d-8)}{2}$ imply

$$(2Z - D)\cdot Z = (d-3)(d-6), \quad (3Z-2D)\cdot(2Z-D) = (d-9)(d-6). \quad (33)$$

By Lemma 1, we have the following two cases.

case (1): $|\sigma Z - (\sigma-2)D| \neq \emptyset$.

In this case, from

$$\alpha Z + \beta(3Z - 2D) = \sigma Z - (\sigma-2)D,$$

we obtain

$$\alpha = \frac{6-\sigma}{2}, \beta = \frac{\sigma-2}{2}.$$

Since $2Z - D$ is nef for $\sigma \geq 4$, it follows that

$$(\sigma Z - (\sigma-2)D)\cdot(2Z-D) \geq 0.$$

Hence,

$$\begin{aligned}
&(\sigma Z - (\sigma-2)D)\cdot(2Z-D)\\
&= (\alpha Z + \beta(3Z-2D))\cdot(2Z-D)\\
&= \alpha Z\cdot(2Z-D) + \beta(3Z-2D)\cdot(2Z-D)\\
&= \alpha(d-3)(d-6) + \beta(d-6)(d-9)\\
&= \frac{6-\sigma}{2}(d-3)(d-6) + \frac{\sigma-2}{2}(d-6)(d-9)\\
&\geq 0.
\end{aligned}$$

By $d > 6$, we obtain $2d - 3\sigma \geq 0$. Hence, $\sigma \leq \frac{2d}{3}$.

By hypothesis, $\sigma \geq d-2$. Thus $\frac{2d}{3} \geq d-2$,which induces $d \leq 6$. This contradicts the hypothesis that $d > 6$.

case (2): $B = 1, 2f < \sigma$ and $|eZ-(e-3)D| \neq \emptyset$.
Then solving the following equation:

$$\alpha Z + \beta(3Z-2D) = eZ-(e-3)D,$$

we obtain

$$\alpha = \frac{9-\sigma}{2}, \beta = \frac{e-3}{2}.$$

Since $2Z-D$ is nef for $\sigma \geq 4$, it follows that

$$(eZ-(e-3)D)\cdot(2Z-D) \geq 0.$$

By the same argument as before, we conclude that $d \geq e$.

But by hypothesis,$\sigma \geq d-2$.

On the other hand, $e = f+\sigma \geq \nu_1+\sigma$. Thus $d \geq e \geq \nu_1+\sigma$; thus $\sigma \geq d-2 \geq \nu_1+\sigma-2$. Hence, $\nu_1 = 1, 2$.

If $\nu_1 = 1$ then $f \geq 2$ by # minimality and hence, $e-\sigma = 2$ and $f = 2$. The type becomes $[\sigma*(\sigma+2), 1; 1]$. Contracting Δ_∞ into a point, we have a singular plane curve with only one double point.

If $\nu_1 = 2$ then $e-\sigma = 2, f = 2$. In this case, The type becomes $[\sigma*(\sigma+2), 1; 2^r]$. Contracting Δ_∞ into a point, we have a singular plane curve with $r+1$ double points.

Apart from this case , we have $d \geq 2\sigma-1$.

7.2 Numerical examples

Table 15: types in which $P_{2,1}[D] \geq \frac{(d-4)(d-5)}{2}$ and $P_{3,1}[D] = \frac{(d-7)(d-8)}{2}$ with $10 \leq d \leq 21$ and $\Delta_{21} = t_2 = 0$

d	$[\sigma * e\ ,\ B$;multiplicities$]$	ν_1
10	$[6 * 8; 1]$	1
11	$[6 * 11; 3^3]$	3
12	$[6 * 15; 3^8]$	3
12	$[7 * 10; 1]$	1
13	$[6 * 20; 3^{15}]$	3
13	$[7 * 16, 1; 3^2]$	3
14	$[6 * 26; 3^{24}]$	3
14	$[7 * 19, 1; 3^5]$	3
14	$[8 * 12; 1]$	1
15	$[6 * 33; 3^{35}]$	3
15	$[7 * 19; 3^9]$	3
16	$[6 * 41; 3^{48}]$	3
16	$[7 * 23; 3^{14}]$	3
16	$[8 * 17; 3^4]$	3
16	$[8 * 18; 4^3]$	4
16	$[9 * 14; 1]$	1
17	$[6 * 50; 3^{63}]$	3
17	$[7 * 31, 1; 3^{20}]$	3
17	$[8 * 20; 3^7]$	3
17	$[8 * 21; 4^3, 3^3]$	4
17	$[9 * 21, 1; 4]$	4

Table 16: types in which $P_{2,1}[D] \geq \frac{(d-4)(d-5)}{2}$ and $P_{3,1}[D] = \frac{(d-7)(d-8)}{2}$ with $18 \leq d \leq 20$, $t_2 = 0$

d	[$\sigma * e$, B;multiplicities]	ν_1
18	$[6 * 60; 3^{80}]$	3
18	$[7 * 36, 1; 3^{27}]$	3
18	$[8 * 24; 4^2, 3^8]$	4
18	$[8 * 25; 4^5, 3^4]$	4
18	$[8 * 26; 4^8]$	4
18	$[9 * 19; 4, 3^2]$	4
18	$[10 * 16; 1]$	1
19	$[6 * 71; 3^{99}]$	3
19	$[7 * 38; 3^{35}]$	3
19	$[8 * 27; 3^{15}]$	3
19	$[8 * 28; 4^3, 3^{11}]$	4
19	$[8 * 29; 4^6, 3^7]$	4
19	$[8 * 30; 4^9, 3^3]$	4
19	$[9 * 26, 1; 3^6]$	3
19	$[9 * 22; 4^2, 3^3]$	4
19	$[9 * 27, 1; 4^4]$	4
20	$[6 * 83; 3^{120}]$	3
20	$[7 * 44; 3^{44}]$	3
20	$[8 * 31; 3^{20}]$	3
20	$[8 * 32; 4^3, 3^{16}]$	4
20	$[8 * 33; 4^6, 3^{12}]$	4
20	$[8 * 34; 4^9, 3^8]$	4
20	$[8 * 35; 4^{12}, 3^4]$	4
20	$[8 * 36; 4^{15}]$	4
20	$[9 * 29, 1; 3^9]$	3
20	$[9 * 25; 4^2, 3^6]$	4
20	$[9 * 30, 1; 4^4, 3^3]$	4
20	$[9 * 26; 4^6]$	4
20	$[10 * 21; 4^2]$	4
20	$[11 * 18; 1]$	1

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