RELATIONSHIPS BETWEEN BIRATIONAL INVARIANTS ω AND σ OF ALGEBRAIC PLANE CURVES

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Contents

1. Introduction

Here, we shall study birational properties of algebraic plane curves from the viewpoint of Cremonian geometry. As a matter of fact, let S be a nonsingular rational surface and D a nonsingular curve on S. (S, D) are called pairs and we study such pairs. The purpose of Cremonian geometry is the study of birational properties of pairs (S, D).

Suppose that $m \geq a \geq 1$. Then $P_{m,a}[D] = \dim |mK_S + aD| + 1$ are called mixed plurigenera, which depend on S and D. It is my understanding that these invarints embody the essential geometric properties of the curve D on S.

Letting Z stand for $K_S + D$, we see $P_{m,m}[D] = \dim |mZ| + 1$, called *logarithmic plurigenera* of $S - D$, from which logarithmic Kodaira dimension is introduced, denoted by $\kappa[D]$.

Assume that $\kappa[D] = 2$ and that there exist no (-1) curves E such that $E \cdot D \leq 1$. Then such pairs are proved to be minimal models in the birational geometry of pairs ([7],[6]).

Moreover, if $S \neq \mathbf{P^2}$, then there exists a surjective morphism $pr : S \to \mathbf{P^1}$ whose general fibers are $\mathbf{P^1}$. The least mapping degree of $pr|_D : D \to \mathbf{P^1}$ for all such pr,is denoted by σ.

By definition, $P_{1,1}[D] = g$, which is the genus of D, and $\overline{g}$ is defined to be $g - 1$.

If $\sigma > 4$ then $D + 2K_S$ is nef and big; furthermore,
$P_{2,1}[D] = Z^2 - \overline{g} + 1 = A + 1$, where $A = Z^2 - \overline{g}$;

If $\sigma > 6$ then $|D + 3K_S| \neq \emptyset$ and

$$P_{3,1}[D] = 3Z^2 + 1 - 7\overline{g} + D^2 = 3A - \alpha + 1 = \Omega - \omega + 1$$

where $\alpha = 4\overline{g} - D^2, \Omega = (3Z - 2D) \cdot Z = 3Z^2 - 4\overline{g}$ and $\omega = 3\overline{g} - D^2$.

The invariant ω is rewritten as $\frac{(D+3K_S)\cdot D}{2}$.

2. MAIN RESULT

ω is a very powerful invariant, which determines the basic structure of pairs (S, D) . We shall establish the upper bound estimate of σ by ω provided that $\kappa[D] = 2$, (S, D) is minimal and $\sigma \geq 7$. Namely, we shall verify the next inequality (1).

Theorem 1. *If $\sigma \geq 7$ then*

$$\sigma \leq (\omega + 1)(\omega + 2) = \omega^2 + 3\omega + 2 \tag{1}$$

except for the type $[7 * 9, 1; 1]$.

In the exceptional case, $\sigma = 7$ and $\omega = 1$.

2.1. minimal models.

We start with recalling some basic results in birational geometry of pairs.

Proposition 1. *Suppose that (S, D) is minimal. Let g denote the genus of the curve D.*

(1) *If $g > 0$ then $Z = K_S + D$ is nef. Moreover, when $\kappa[D] = 2$, Z is big.*
(2) *If $g = 0$ and $\kappa[D] = 2$ then $D^2 \leq -5$ and letting β denote $-D^2$, $Z_\beta = Z - \frac{2}{\beta}D$ is nef and big.*

Minimal pairs are obtained from some kind of singular models, namely, # minimal pairs which will be defined below. Any nontrivial $\mathbf{P}^1-$ bundle over $\mathbf{P}^1$ has a section Δ_∞ with negative self intersection number, which is denoted by a symbol Σ_B, where $-B = {\Delta_\infty}^2$ if $B > 0$. Σ_B is said to be a Hirzebruch surface of degree B after Kodaira.

Let Σ_0 denote the product of two projective lines.

The Picard group of Σ_B is generated by a section Δ_∞ and a fiber $F_c = \rho^{-1}(c)$ of the $\mathbf{P}^1-$ bundle, where $c \in \mathbf{P}^1$ and $\rho : \Sigma_B \to \mathbf{P}^1$ is the projection.

Let C be an irreducible curve on Σ_B. Then $C \sim \sigma\Delta_\infty + eF_c$, for some σ and e. Here the symbol $\sim$ means the linear equivalence between divisors. We have $C \cdot F_c = \sigma$ and $C \cdot \Delta_\infty = e - B \cdot \sigma$. Note that $\kappa[\Delta_\infty] = -\infty$.

Hereafter, suppose that $C \neq \Delta_\infty$. Thus $C \cdot \Delta_\infty = e - B \cdot \sigma \geq 0$ and hence, $e \geq B\sigma$. Denoting $2e - B\sigma$ by $\widetilde{B}$, we have the formula:

$$g_0 = \frac{(\sigma - 1)(\widetilde{B} - 2)}{2}.$$

Thus introducing τ_m by

$$\tau_m = (\sigma - m)(\widetilde{B} - 2m), \tag{2}$$

we obtain

$$(K_0 + C)^2 = \tau_2,$$

where K_0 denotes a canonical divisor on Σ_B.

Moreover, letting Z_0 be $K_0 + C$, we obtain

$$\nu Z_0 - (\nu - 1)C \sim C + \nu K_0$$
$$(\nu Z_0 - (\nu - 1)C) \cdot Z_0 = \tau_{\nu+1} - 2(\nu - 1)^2,$$

and

$$(\nu Z_0 - (\nu - 1)C) \cdot C = \tau_\nu - 2\nu^2.$$

Furthermore, define ω_0 by

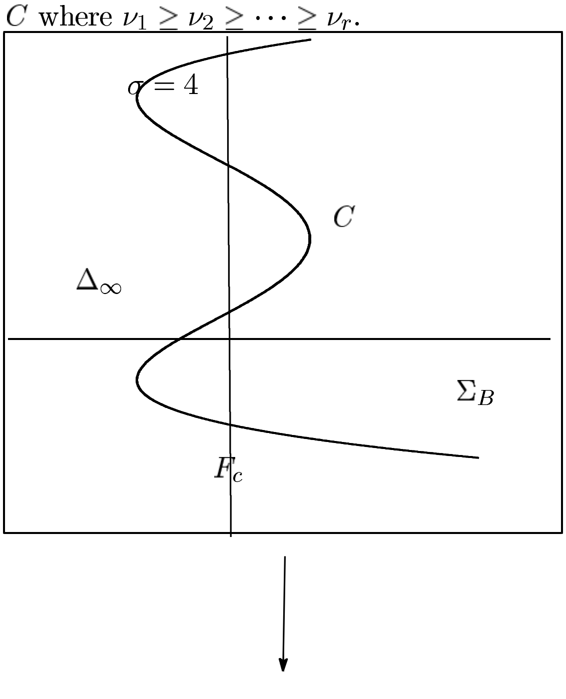
$$\omega_0 = \frac{(3Z_0 - 2C) \cdot C}{2}.$$

Then $\omega_0 = \frac{\tau_3}{2} - 9$.

Therefore,

$$\begin{aligned}\omega &= \omega_0 - \sum_{j=1}^{r} \frac{\nu_j(\nu_j - 3)}{2} \\ &= \frac{\tau_3}{2} - 9 - \sum_{j=1}^{r} \frac{\nu_j(\nu_j - 3)}{2}.\end{aligned}$$

2.2. **types of pairs and # minimal pairs.** By $\nu_1, \nu_2, \cdots, \nu_r$ we denote the multiplicities of all singular points (including infinitely near singular points) of C where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$.



The symbol $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$ is said to be the **type** of (Σ_B, C).

Definition 1. the pair (Σ_B, C) is said to be # **minimal** , if

- $\sigma \geq 2\nu_1$ and $e - \sigma \geq B\nu_1$;
- moreover, if $B = 1$ and $r = 0$ then assume $e - \sigma > 1$.

Using elementary transformations, we get

Theorem 2. *If D is not transformed into a line on $\mathbf{P^2}$ by Cremona transformations, then $\kappa[D] \geq 0$. In this case, a minimal pair (S, D) is obtained from a # minimal pair (Σ_B, C) by shortest resolution of singularities of C using blowing ups except for $(S, D) = (\mathbf{P^2}, C_d)$, C_d being a nonsingular curve.*

Theorem 3. *If (S, D) is obtained from a # minimal pair(model) (Σ_B, C) by shortest resolution of singularities of C ,then (S, D) is relatively minimal. In other words,for any (-1) curve Γ on S, $\Gamma \cdot \Delta \geq 2$.*

2.3. ω **for nonsingular plane curves.**
First we treat pairs with small σ(cf Terashima([10])). Suppose that D is a nonsingular plane curve on $\mathbf{P^2}$ of degree d. Then $\omega = \frac{d(d-9)}{2}$ and we obtain the next table.

TABLE 1. ω for nonsingular plane curves

d	3	4	5	6	7	8	9	10
ω	-9	-10	-10	-9	-7	-4	0	5

2.4. ω **for small** σ**.**

2.4.1. $\sigma = 2$.
Suppose that $\sigma = 2$. Then $D^2 = 4g+4 = 4\overline{g}+8$ and hence, $\omega = 3\overline{g} - D^2 = -7 - g \leq -8$ and $\alpha = -8$.

2.4.2. $\sigma = 3$.
Suppose that $\sigma = 3$. Then $D^2 = 3g + 6 = 3\overline{g} + 9$ and hence, $\omega = -9$ and $\alpha = \overline{g} - 9$.

2.4.3. $\sigma = 4$.
Suppose that $\sigma = 4$. Then $\widetilde{B} = 2e - 4B$. We distinguish the various cases according to the value of B.

(1) $B = 0$. Then $e = 4 + u$ and $\widetilde{B} = 2(4 + u) = 8 + 2u$; thus, $\tau_3 = 2(u + 1)$ and so $\omega = u + 1 - 9 + t_2 = u + t_2 - 8$. Since $g = 9 + 3u - t_2 \geq 0$, we get $t_2 \leq 9 + 3u$.

(2) $B = 1$. Then $e = 4 + u + \nu_1$ and $\widetilde{B} = 4 + 2u + 2\nu_1$; thus, $\tau_3 = 2u + 2\nu_1 - 2$. Hence, $\omega = u + \nu_1 - 10 + t_2$. Since $g = 3(1 + u + \nu_1) - t_2 \geq 0$, we get $t_2 \leq 3 + 3\nu_1 + 3u$.

If $r = 0$ then $u \geq 1$ and $\omega = u - 9 \geq -8$.

If $\nu_1 = 2$ then $t_2 \geq 1$ and $\omega = u - 8 + t_2 \geq -7$.

TABLE 2. ω when $\sigma = 4, B = 0$

u / t_2	0	1	2	3	4	5	6	7	8
0	−8	−7	−6	−5	−4	−3	−2	−1	0
1	−7	−6	−5	−4	−3	−2	−1	0	1
2	−6	−5	−4	−3	−2	−1	0	1	2
3	−5	−4	−3	−2	−1	0	1	2	3

TABLE 3. ω when $\sigma = 4, B = 1, r = 0$

u	1	2	3	4	5	6	7	8
ω	−8	−7	−6	−5	−4	−3	−2	−1

TABLE 4. ω when $\sigma = 4, B = 1, \nu_1 = 2$

u / t_2	0	1	2	3	4	5	6	7	8
1	−7	−6	−5	−4	−3	−2	−1	0	1
2	−6	−5	−4	−3	−2	−1	0	1	2
3	−5	−4	−3	−2	−1	0	1	2	3

(3) $B \geq 2$. Then $e = 4B + u$ and $\widetilde{B} = 4B + 2u$; thus, $\tau_3 = 4B - 6 + 2u$ and so $\omega = 2(B-2) + u - 8 + t_2$.

If $r = 0$, then $\omega = 2(B-2) + u - 8$. Otherwise, $\omega = 2(B-2) + u - 8 + t_2$.

TABLE 5. ω when $\sigma = 4, B \geq 2, r = 0$

u / B	0	1	2	3	4	5	6	7	8
2	−8	−7	−6	−5	−4	−3	−2	−1	0
3	−7	−6	−5	−4	−3	−2	−1	0	1
4	−6	−5	−4	−3	−2	−1	0	1	2

2.4.4. $\sigma = 5$.

Suppose that $\sigma = 5$. Then $\widetilde{B} = 2e - 5B$. We distinguish the various cases according to the value of B.

(1) $B = 0$. Then $e = 5 + u$ and $\widetilde{B} = 10 + 2u$; thus, $\tau_3/2 = 2u - 1$ and so $\omega = 2u - 5 + t_2$. Since $g = 4(4+u) - t_2 \geq 0$, we get $t_2 \leq 16 + 4u$.

(2) $B = 1$. Then $e = 5 + u + \nu_1$ and $\widetilde{B} = 5 + 2u + 2\nu_1$; thus, $\tau_3 = 2(2u + 2\nu_1 - 1)$. Hence, $\omega = 2u + 2\nu_1 - 10 + t_2$.

If $r = 0$ then $u \geq 1$ and $\omega = 2u - 8$.

If $r > 0$ then $\omega = 2u - 6 + t_2$.

TABLE 6. ω when $\sigma = 5, B = 0$

u \ t_2	0	1	2	3	4	5
0	−5	−3	−1	1	3	5
1	−4	−2	0	2	4	6
2	−3	−1	1	3	5	7
3	−2	0	2	4	6	8

TABLE 7. ω when $\sigma = 5, B = 1, r = 0$

u	1	2	3	4	5	6	7
ω	−6	−4	−2	0	2	4	6

TABLE 8. ω when $\sigma = 5, B = 1, r > 0$

u \ t_2	0	1	2	3	4
1	−5	−3	−1	1	3
2	−4	−2	0	2	4
3	−3	−1	1	3	5

Since $g = 3(1 + u + \nu_1) - t_2 \geq 0$, we get $t_2 \leq 3 + 3\nu_1 + 3u$.
If $r = 0$ then $u \geq 1$ and $\omega = u - 9$.
If $\nu_1 = 2$ then $t_1 \geq 1$ and $\omega = u - 8 + t_2$.

2.4.5. $\sigma = 6$.

Suppose that $\sigma = 6$. Then $\widetilde{B} = 2e - 6B$. We distinguish the various cases according to the value of B.

(1) $B = 0$. Then $e = 6 + u$ and $\widetilde{B} = 12 + 2u$; thus, $\tau_3/2 = 9 + 3u$ and so $\omega = 3u + t_2$. Since $g = 25 + 5u - t_2 - 3t_3 \geq 0$, we get $t_2 + 3t_3 \leq 25 + 5u$.

TABLE 9. ω when $\sigma = 6, B = 0$

u \ t_2	0	1	2	3	4	5
0	0	3	6	9	12	
1	1	4	7	10	13	

(2) $B = 1$. Then $e = 6 + u + \nu_1$ and $\widetilde{B} = 6 + 2u + 2\nu_1$; thus, $\tau_3 = 6(u + \nu_1)$. Hence, $\omega = 3u + 3\nu_1 - 9 + t_2$.
If $r = 0$ then $u \geq 1$ and $\omega = 3u - 6$.
If $r > 0$ and $t_3 = 0$ then $e = 8 + u$ and $\omega = 3u - 3 + t_2$.
If $r > 0$ and $t_3 > 0$ then $e = 9 + u$ and $\omega = 3u + t_2$.

TABLE 10. ω when $\sigma = 6, B = 1, r = 0$

u	1	2	3	4	5	6	7
	-3	0	3	6	9	12	15

TABLE 11. ω when $\sigma = 6, B = 1, r > 0$

u	0	1	2	3	4
t_2					
1	-2	1	4	7	10
2	-1	2	5	8	11
3	0	3	6	9	12

TABLE 12. ω when $\sigma = 6, B = 1, t_3 > 0$

u	0	1	2	3	4
t_2					
0	3	6	9	12	15
1	4	7	10	13	16
2	5	8	11	14	17
3	6	9	12	15	18

2.4.6. $\sigma = 7$.

Suppose that $\sigma = 7$. Then $\widetilde{B} = 2e - 7B$. We distinguish the various cases according to the value of B.

(1) $B = 0$. Then $e = 7 + u$ and $\widetilde{B} = 14 + 2u$; thus, $\tau_3/2 = 16 + 4u$ and so $\omega = 7 + 4u + t_2 \geq 7$.

TABLE 13. ω when $\sigma = 7, B = 0$

u	0	1	2	3	4	5
t_2						
0	7	11	15	16	21	
1	8	12	16	17	22	

(2) $B = 1$. Then $e = 7 + u + \nu_1$ and $\widetilde{B} = 7 + 2u + 2\nu_1$; thus, $\tau_3 = 4(1 + 2u + 2\nu_1)$. Hence, $\omega = 2 + 4u + 4\nu_1 - 9 + t_2$.

If $r = 0$ then $u \geq 1$ and $\omega = 4u - 3$.

If $r > 0$ and $t_2 > 0, t_3 = 0$ then $\omega = 4u + 1 + t_2$.

If $r > 0$ and $t_2 \geq 0, t_3 > 0$ then $e = 10 + u$ and $\omega = 4u + 5 + t_2$.

TABLE 14. ω when $\sigma = 7, B = 1, r = 0$

u	1	2	3	4	5	6	7
ω	1	5	9	13	17	21	25

TABLE 15. ω when $\sigma = 7, B = 1$

u	0	1	2	3	4
t_2					
1	2	6	10	14	18
2	3	7	11	15	19
3	4	8	12	16	20

2.5. graph.

The following figure is obtained by plotting (ω, σ).

Note that in Figure 1,there are many parabolas; these are defined by $y = x^2 + 3x + 2, y = x^2 + x + 2, y = x^2 - x + 4, y = x^2 - x + 2,$

FIGURE 1

We have the following inequality, which is closely related to the inequality (1).

Theorem 4. *Let $\overline{g}$ denote $g - 1$. If $\sigma \geq 7$, then*

$$\sigma \leq \omega_1^2 + \omega_1 + 2 + 2\overline{g}. \tag{3}$$

Here, $\omega_1 = \omega - \overline{g}$ and $\omega_1 = K_S \cdot D$.

- The domain (I) is defined by $y \leq x^2 + 3x + 2, y > x^2 + x + 2$.
- The domain (II) is defined by $y \leq x^2 + x + 2, y > x^2 - x + 4$.

2.6. proof of the inequality (1) in the case when $\nu_1 \leq 3$.

First, we consider the case when $\nu_1 \leq 3$.

By the formula, we get $\omega = \tau_3/2 - 9 + t_2 \geq \tau_3/2 - 9$.

Assuming $\sigma \geq 7$, we distinguish the various cases according to the value of B.

(1) If $B=0$, then $e \geq \sigma$ and

$$\tau_3 = (\sigma-3)(2e-6) \geq 2(\sigma-3)^2 \geq 8(\sigma-3).$$

Hence,

$$\omega \geq \tau_3/2 - 9 \geq 4(\sigma-3) - 9 = 4\sigma - 21 \geq 7.$$

Thus,

$$\sigma \leq \frac{\omega+21}{4}.$$

In particular,

$$\sigma < (\omega+1)(\omega+2).$$

(2) If $B=1$, then $e-\sigma \geq 2$ and $\widetilde{B}-6 = 2e-\sigma-6 \geq e-4 \geq \sigma-2$.
Hence,

$$\tau_3 = (\sigma-3)(\widetilde{B}-6) \geq 5(\sigma-3).$$

Thus

$$\omega \geq \omega_0 = \frac{\tau_3}{2} - 9 \geq \frac{5\sigma-33}{2}.$$

Therefore,

$$\sigma \leq \frac{2\omega+33}{5}.$$

If $\omega = 1$ then $\sigma \leq 7$. Assume further that $\sigma = 7$. Then $e=9$ and the type is $[7*9,1;1]$.
Moreover, $\omega^2+3\omega+2-(\frac{2\omega+33}{5}) = \frac{5\omega^2+13\omega-23}{5} > 0$ for $\omega > 1$.
Hence,

$$\omega^2+3\omega+2 \geq (\frac{2\omega+33}{5}) > \sigma.$$

(3) If $B \geq 2$, then $e-2\sigma = e-B\sigma+(B-2)\sigma \geq (B-2)\sigma \geq 0$ and so $\widetilde{B}-6 = 2e-B\sigma-6 \geq e-6 \geq 2\sigma+(B-2)\sigma-6$. Hence,

$$\tau_3 = (\sigma-3)(\widetilde{B}-6) \geq 2(\sigma-3)(\sigma-3)+(B-2)\sigma(\sigma-3) \geq 8(\sigma-3),$$

and moreover,

$$\omega \geq \tau_3/2 - 9 \geq 4(\sigma-3) - 9 = 4\sigma - 21 \geq 7.$$

Thus,

$$\sigma \leq \frac{\omega+21}{4}.$$

In particular,

$$\sigma < (\omega+1)(\omega+2).$$

2.6.1. *examples.*

If the type is $[7*9,1;2^r]$ then $g=27-r, D^2=77-4r, \omega=1+r$.

If the type is $[8*12,1;4^r]$ then $g=49-6r, D^2=128-16r, \omega=16-2r$.

When $r=8$, we get $\kappa[D]=1$. This contradicts the hypothesis that $\kappa[D]=2$.

When $r=7$, $\kappa[D]=2$ and $\omega=2$.

When $r=6$, we have $\kappa[D]=2$ and $\omega=4$.

Moreover, if the type is $[8*12,1;4^7,3^{t_3}]$ where $t_3=1,2$, then $\omega=2$.

If the type is $[8*12,1;4^7,3^{t_3},2]$ where $t_3=1,2$, then $\omega=3$.

Finally, if the type is $[8*12,1;4^6,3^{t_3}]$ where $t_3=1,2,3,4$, then we have $\omega=4$.

2.7. **proof of the inequality (2) in the case when $r=0$.**

Assume that $r=0$. Then $\omega=\frac{\tau_3}{2}-9$ and $\overline{g}=\frac{\tau_1}{2}-1$. Hence,

$$\omega_1=\omega-\overline{g}=\frac{\tau_3-\tau_1}{2}-8.$$

But, $\tau_3-\tau_1=-2B'+16$, where $B'=2\sigma+\widetilde{B}$. It is easy to check

- if $B\neq 1$,then $B'\geq 4\sigma$;
- if $B=1$,then $B'\geq 3\sigma$.

By using B', we have $\omega_1=-B'$, $\omega_1(\omega_1+1)=B'(B'-1)$ and $2\overline{g}=\sigma\widetilde{B}-B'$.
Thus,

$$\omega_1^2+\omega_1+2+2\overline{g}=(B'-1)^2+1+\sigma\widetilde{B}\geq 5\sigma^2.$$

2.8. **lemma.** We shall use the next lemma:

Lemma 1. *Suppose that (S,D) is minimal.*

1. *If $B=0$ or 2 then $2\sigma\overline{g}-(\sigma-2)D^2=(2D+\sigma K_S)\cdot D\geq 0$.*
2. *If $B=1$ then either $(2D+\sigma K_S)\cdot D\geq 0$ or $(3D+eK_S)\cdot D\geq 2$.*
3. *If $B\geq 2$ then $(2D+\sigma K_S)\cdot D\geq (e+e-\sigma B-2\sigma)\sigma\geq \sigma^2(B-2)$.*
4. *If $B\geq 3$ then $(2D+\sigma K_S)\cdot D\geq \sigma^2$; in particular, $2\sigma\overline{g}-(\sigma-2)D^2\geq\sigma^2$. Hence, $\sigma\omega_1+2D^2\geq\sigma^2$.*

Proof. From $\sigma K_0+2C\sim(2e-\sigma(B+2))F_c$, it follows that

$$\begin{aligned}(2D+\sigma K_S)\cdot D&=(2C+\sigma K_0)\cdot C+\sum_{j=1}^{r}(\sigma-2\nu_j)\nu_j\\&=(e+e-\sigma B-2\sigma)\sigma+pY+2\widetilde{\mathcal{Z}}\\&\geq\sigma^2(B-2).\end{aligned}$$

where $X=\sum_{j=1}^r\nu_j^2$ $Y=\sum_{j=1}^r\nu_j$ and $\mathcal{Z}^*=\nu_1Y-X$. Thus,

$$(\sigma K_S+2D)\cdot D\geq(\sigma K_0+2C)\cdot C\geq\sigma^2(B-2).$$

From this one can verify (1),(3) and (4).

As for (1), when $B=1$, if $(2C+\sigma K_0)\cdot C=(2e-3\sigma)\sigma<0$ then

$$\begin{aligned}(3D+eK_S)\cdot D &= (3C+eK_0)\cdot C+\sum_{j=1}^{r}(e-3\nu_j)\nu_j\\ &= (3C+eK_0)\cdot C+(p+u)Y+3\widetilde{\mathcal{Z}}\\ &= (3\sigma-2e)\Delta_\infty\cdot(\sigma\Delta_\infty+eF_c)+(p+u)Y+3\widetilde{\mathcal{Z}}\\ &= (3\sigma-2e)(e-\sigma)+(p+u)Y+3\widetilde{\mathcal{Z}}\\ &= (3\sigma-2e)(u+\nu_1)+(p+u)Y+3\widetilde{\mathcal{Z}}\\ &\geq u+\nu_1\geq 2.\end{aligned}$$

Here note that $e-3\nu_j=\sigma+u+\nu_1-3\nu_j=\sigma-2\nu_j+u+\nu_1-\nu_j\geq 0$. Q.E.D.

2.9. **proof of the inequality (1) in the case when $B\geq 3$.** Suppose that $B\geq 3$.

Then by Lemma 1(3),

$$\sigma\omega_1\geq -2D^2+\sigma^2,$$

in other words,

$$\omega_1\geq \sigma-\frac{2}{\sigma}D^2.$$

If $D^2\leq 0$, then

$$\omega_1\geq \sigma-\frac{2}{\sigma}D^2\geq\sigma.$$

Hence,

$$\omega=\omega_1+\overline{g}\geq\sigma-1.$$

Therefore,

$$\sigma\leq\omega+1.$$

If $D^2>0$, then since $\omega=3\overline{g}-D^2\geq 0$, it follows that $3\overline{g}\geq D^2$ and so

$$\omega=\omega_1+\overline{g}\geq\sigma+\frac{\sigma-6}{\sigma}D^2\geq\sigma.$$

Thus $\sigma\leq\omega$. Q.E.D.

In particular,

$$\sigma<(\omega+1)(\omega+2).$$

2.10. **proof of the inequality (2) in the case when $B\geq 3$.**

By Lemma 1 (3), $2\sigma\overline{g}-(\sigma-2)D^2\geq\sigma^2$. From this, it follows that

$$2\overline{g}-(\frac{\sigma-2}{\sigma})D^2\geq\sigma.$$

Replacing D^2 by $\omega_1+2\overline{g}$, we obtain

$$\frac{4\overline{g}}{\sigma}+\frac{\sigma-2}{\sigma}\omega_1\geq\sigma.$$

We distinguish the various cases according to the signature of ω_1.

(1) Suppose that $\omega_1 \geq 0$.
If $\overline{g} = -1$ then $\omega_1 > \sigma$, and hence,

$$\omega_1^2 + \omega_1 + 2 + 2\overline{g} = \omega_1(\omega_1 + 1) > \sigma^2 + \sigma > 49 + \sigma.$$

If $\overline{g} \geq 0$ then $\frac{4\overline{g}}{7} + \omega_1 > \sigma$.
From this, we get

$$\omega_1^2 + \omega_1 + 2 + 2\overline{g} - (\frac{4\overline{g}}{7} + \omega_1) = \omega_1^2 + 2 + \frac{10\overline{g}}{7} \geq 2 > 0.$$

Thus the inequality (2) is obtained.

(2) Suppose that $\omega_1 \leq 0$. Then

$$\frac{4\overline{g}}{7} \geq \sigma,$$

thus

$$\omega_1^2 + \omega_1 + 2 + 2\overline{g} - \frac{4\overline{g}}{7} = \omega_1^2 + \omega_1 + 1 + \frac{10\overline{g}}{7} + 1 > 0.$$

Hence,

$$\begin{aligned} \omega_1^2 + \omega_1 + 2 + 2\overline{g} &\geq \frac{4\overline{g}}{7} \\ &> \sigma. \end{aligned}$$

In that follows, we suppose that $B \leq 2$.

3. FUNDAMENTAL EQUALITIES

By

$$2\omega = (D + 3K_S) \cdot D, 2\omega_0 = (C + 3K_0) \cdot C,$$

and

$$2\overline{g} = (D + K_S) \cdot D, 2\overline{g}_0 = (C + K_0) \cdot C,$$

we get

- $\omega = \omega_0 - \sum_{j=1}^{r} \dfrac{\nu_j(\nu_j - 3)}{2}$,
- $g = g_0 - \sum_{j=1}^{r} \dfrac{\nu_j(\nu_j - 1)}{2}$.

By putting $X = \sum_{j=1}^{r} \nu_j^2$ and $Y = \sum_{j=1}^{r} \nu_j$, we obtain

- $2\omega - 2\omega_0 = -X + 3Y$,
- $2g - g_0 = -X + Y$.

Thus

- $X = 3g_0 - \omega_0 - 3g + \omega$,
- $Y = g_0 - \omega_0 - g + \omega$.

However, from $\omega_0 = \frac{\tau_3}{2} - 9$ and $\overline{g_0} = \frac{\tau_1}{2} - 1$, it follows that

- $\overline{g_0} - \omega_0 = \widetilde{B} + 2\sigma$,
- $3\overline{g_0} - \omega_0 = \widetilde{B}\sigma$.

Consequently we obtain the next equalities:

- $X = \widetilde{B}\sigma + \omega - 3\overline{g}$,
- $Y = \widetilde{B} + 2\sigma + \omega - \overline{g}$.

3.1. two invariants.

We shall compute two invariants $\widetilde{B} + 2\sigma$ and $\widetilde{B}\sigma$ by examining the following cases according to the value of B.

(1) $B = 0$. Then $\sigma = 2\nu_1 + p, e = \sigma + u$ for some $u \geq 0$ and

- $\widetilde{B} + 2\sigma = 8\nu_1 + 4p + 2u$,
- $\widetilde{B}\sigma = 8\nu_1^2 + 2\nu_1(4p + 2u) + 2pu + 2p^2$.

(2) case $B = 1$. Then $\sigma = 2\nu_1 + p, e = \sigma + \nu_1 + u$ for some $u \geq 0$ and

- $\widetilde{B} + 2\sigma = 8\nu_1 + 3p + 2u$,
- $\widetilde{B}\sigma = 8\nu_1^2 + 2\nu_1(3p + 2u) + 2pu + p^2$.

(3) $B = 2$. Then $\sigma = 2\nu_1 + p, e = 2\sigma + u$ for some $u \geq 0$ and

- $\widetilde{B} + 2\sigma = 8\nu_1 + 4p + 2u$,
- $\widetilde{B}\sigma = 8\nu_1^2 + 2\nu_1(4p + 2u) + 2pu + 2p^2$.

Defining $w = 4 - \delta_{1B}$,we get $w = 4$ if $B \neq 1$. Further, $w = 3$ if $B = 1$. Introducing an invariant k by $k = wp + 2u$, we have

- $\widetilde{B} + 2\sigma = 8\nu_1 + k$,
- $\widetilde{B}\sigma = 8\nu_1^2 + 2k\nu_1 + p(k - 2p)$.

Proposition 2. *Letting k denote $wp + 2u$, w being $4 - \delta_{1B}$, we have the following fundamental equalities:*

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $Y = 8\nu_1 + k + \omega_1$.

3.2. invariant $\widetilde{\mathcal{Z}}$.

Following Matsuda([9]), we shall compute $\nu_1 Y - X$, which we denote by $\widetilde{\mathcal{Z}}$.

By $\widetilde{\mathcal{Z}} = \nu_1 Y - X = \sum_{j=1}^{r} \nu_j(\nu_1 - \nu_j) \geq 0$, we have

$$0 \leq \widetilde{\mathcal{Z}} = \nu_1(\omega - \overline{g} - k) - \tilde{k} - \omega_1 + 2\overline{g}. \qquad (4)$$

Here $\tilde{k} = kp - 2p^2$.

Defining the invariant λ to be $k - \omega_1$, we obtain

$$0 \leq \nu_1 Y - X = -\nu_1\lambda - \tilde{k} - \omega_1 + 2\overline{g}.$$

Hence,

$$\nu_1\lambda \leq -\tilde{k} - \omega_1 + 2\overline{g}. \tag{5}$$

3.3. **case in which $B \geq 3$.**

In the case when $B > 2$, by B_2 we denote $B - 2$. Then $e = B\sigma + u = B_2\sigma + 2\sigma + u$ for some $u \geq 0$ and $\widetilde{B} = 2e - B\sigma = B_2\sigma + 2(\sigma + u)$.

Moreover, $\widetilde{B}\sigma = B_2\sigma^2 + 2(\sigma + u)\sigma$ and so

- $\widetilde{B} + 2\sigma = B_2\sigma + 8\nu_1 + k$,
- $\widetilde{B}\sigma = B_2\sigma^2 + 8\nu_1^2 + 2k\nu_1 + \tilde{k}$.

Thus, we obtain the following fundamental equalities:

- $X = B_2\sigma^2 + 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $Y = B_2\sigma + 8\nu_1 + k + \omega_1$,

where $\omega_1 = \omega - \overline{g}$. Further , we get

$$0 \leq \widetilde{\mathcal{Z}} = B_2\sigma(\nu_1 - \sigma) - k\nu_1 + (\nu_1 - 1)\omega_1 + 2\overline{g} - \tilde{k},$$

and

$$B_2\sigma(\sigma - \nu_1) \leq -k\nu_1 + (\nu_1 - 1)\omega_1 + 2\overline{g} - \tilde{k}.$$

If $B \geq 3$, we have

$$\sigma(\sigma - \nu_1) \leq B_2\sigma(\sigma - \nu_1) \leq -k\nu_1 + (\nu_1 - 1)\omega_1 + 2\overline{g} - \tilde{k}. \tag{6}$$

Hence, the following is derived:

Proposition 3. *If $B \geq 3$,then*

$$2\nu_1^2 \leq \sigma(\sigma - \nu_1) \leq (\nu_1 - 1)\omega_1 + 2\overline{g}.$$

3.3.1. *examples.*

Example 1. *Suppose that the pair has the type $[8 * 24, 3; 4^r]$. Then $\widetilde{B} = 24$ and $g = 77 - 6r$.*

Suppose that $77 - 6r \geq 0$. Then $D^2 = 16(12 - r)$. When $r = 12$, we have $\overline{g} = 4, \omega_1 = 8$ and $\omega = 12$.

3.4. **an estimate of ω.**

Theorem 5. *If $B \geq 3$ and $\nu_1 \geq 4$,then $\omega \geq 12$.*

Proof. Supposing $\omega \leq 11$,we shall derive a contradiction.

From $\overline{g} = \omega - \omega_1$, we get by Proposition 3

$$2\nu_1^2 \leq (\nu_1 - 1)\omega_1 + 2(\omega - \omega_1);$$

thus

$$2\nu_1^2 - (\nu_1 - 3)\omega_1 \leq 2\omega \leq 22.$$

By $\nu_1 \geq 4$, we derive $\omega_1 \geq 10$. Actually, since $\omega_1 > 0$, it follows that

$$10 \leq \frac{2\nu_1^2 - 22}{\nu_1 - 3} \leq \omega_1.$$

Since $\omega \leq 11$ and $\omega_1 \geq 10$, it follows that

$$11 \geq \omega \geq 10 + \overline{g}. \tag{7}$$

Hence, $\overline{g} = 1$ or 0 or -1.

By Lemma 1(4), we have $\sigma\omega_1 + 2D^2 \geq \sigma^2$. Thus

$$\omega_1 \geq \sigma - \frac{2D^2}{\sigma}, \tag{8}$$

which will be used.

We shall distinguish the following cases according to the value of $\overline{g}$.

(1) $\overline{g} = 1$.

Then $\omega = 11$ and $\omega_1 = 10$. But from $\omega_1 = 2\overline{g} - D^2$, we get $D^2 = -8$.

From the inequality (8)

$$10 = \omega_1 \geq \sigma - \frac{2D^2}{\sigma} = \sigma + \frac{16}{\sigma},$$

it follows that $\sigma = 8$.

By making use of the inequality (6) , we obtain $k = 0$. The fundamental formulas turn out to be

- $Y = \sigma + 8\nu_1 + k + \omega_1$,
- $X = \sigma^2 + 8\nu_1^2 + 2k\nu_1 + \omega_1 - 2\overline{g}$.

From these,

$$\widetilde{\mathcal{Z}} = \sigma(\nu_1 - \sigma) + (\nu_1 - 1)\omega_1 + 2\overline{g} = -32 + 30 + 2 = 0.$$

Thus $r = t_4$ and $Y = \sigma + 8\nu_1 + k + \omega_1 = 50 = 4r$, which has no solution.

(2) $\overline{g} = 0$.

Then $\omega = \omega_1 = 10$ or 11.

Assume $\omega = 10$. But from $\omega_1 = 2\overline{g} - D^2$, we get $D^2 = -10$.

From the inequality (6)

$$10 = \omega_1 \geq \sigma - \frac{2D^2}{\sigma} = \sigma + \frac{20}{\sigma},$$

it follows that $\sigma < 8$, a contradiction.

Assume $\omega = 11$. But from $\omega_1 = 2\overline{g} - D^2$, we get $D^2 = -11$.

From the inequality (6)

$$11 = \omega_1 \geq \sigma - \frac{2D^2}{\sigma} = \sigma + \frac{22}{\sigma},$$

it follows that $\sigma = 8$ and $\nu_1 = 4$.

Further,by the inequality (6), we get $k=0$ and so $\widetilde{\mathcal{Z}} = \sigma(\nu_1-\sigma)+(\nu_1-1)\omega_1+2\overline{g} = -32+3\cdot 11 = 1$.

But $\widetilde{\mathcal{Z}} \geq \nu_1 - 1 = 3$, which implies a contradiction.

(3) $\overline{g} = -1$.

Then $\omega_1 = \omega + 1 \geq 10$. Hence, $\omega \geq 9$.

Therefore, $\omega = 9, 10, 11$. Corresponding to these values, we have $D^2 = -12, -13, -14$ since $\omega = -3 - D^2$.

But from $\omega_1 \geq \sigma - \frac{2D^2}{\sigma}$, we get $\omega_1 = 10 \geq \sigma + \frac{24}{\sigma}$ or $\omega_1 = 11 \geq \sigma + \frac{26}{\sigma}$ or $\omega_1 = 12 \geq \sigma + \frac{28}{\sigma}$.

Therefore, we obtain $\sigma = 8$ and $\omega_1 = 12$

Further,by the inequality (6), we get $k=0$ and so $\widetilde{\mathcal{Z}} = \sigma(\nu_1-\sigma)+(\nu_1-1)\omega_1+2\overline{g} = -32+3\cdot 12-2 = 2 \geq \nu\_1 = 3$, a contradiction.

3.5. **case in which** $\omega = 12$**.** Supposing that $B \geq 3$, $\nu_1 \geq 4$ and $\omega = 12$, we shall compute the types.

By $\omega_1 = \omega - \overline{g} = 12 - \overline{g} \leq 13$,

$$2\nu_1^2 - (\nu_1 - 3)\omega_1 \leq 2\omega = 24.$$

By $\nu_1 \geq 4$, we derive $\nu_1 = 4$ and $8 \leq \omega_1 = 12 - \overline{g}$.

Hence, $\overline{g} \leq 4$.

(1) $\overline{g} = 4$. Then $\omega_1 = 8$ and $\omega_1 = 8 - D^2$. Hence, $D^2 = 0$ and

$$8 = \omega_1 \geq \sigma - \frac{2D^2}{\sigma} = \sigma \geq 8.$$

Therefore, $\sigma = 8$ and $k = 0$. Furthermore,

$$\widetilde{\mathcal{Z}} = \sigma(\nu_1 - \sigma) + (\nu_1 - 1)\omega_1 + 2\overline{g} = 0.$$

Then $Y = 8 + 8\cdot 4 + 8 = 48 = 5r$. Hence, $r = 12$ and the type turns out to be $[8 * 24, 3; 4^{12}]$.

(2) $\overline{g} = 3$. Then $\omega_1 = 9$ and $9 = \omega_1 = 6 - D^2$. Hence, $D^2 = -3$. It is easy to derive $k = 0$ and

$$\widetilde{\mathcal{Z}} = \sigma(\nu_1 - \sigma) + (\nu_1 - 1)\omega_1 + 2\overline{g} = 1 \geq \nu_1 - 1 = 3,$$

a contradiction.

(3) $\overline{g} = 2$. Then $\omega_1 = 10$ and $10 = \omega_1 = 4 - D^2$. Hence, $D^2 = -6$. It is easy to derive $k = 0$ and

$$\widetilde{\mathcal{Z}} = \sigma(\nu_1 - \sigma) + (\nu_1 - 1)\omega_1 + 2\overline{g} = 2 \geq \nu_1 - 1 = 3,$$

a contradiction.

(4) $\overline{g} = 1$. Then $\omega_1 = 11$ and $11 = \omega_1 = 2 - D^2$. Hence, $D^2 = -9$. It is easy to derive $k = 0$ and

$$\widetilde{\mathcal{Z}} = \sigma(\nu_1 - \sigma) + (\nu_1 - 1)\omega_1 + 2\overline{g} = 3.$$

Then $\nu_1 = 4$ and $t_3 = 1$. By $Y = 8+8\cdot 4+11 = 51 = 4t_4+3t_3+2t_2 = 4t_4+3$. Hence, $t_4 = 12, r = 13$ and the type turns out to be $[8 * 24, 3; 4^{12}, 3]$.

(5) $\overline{g} = 0$. Then $\omega_1 = 12$ and $12 = \omega_1 = -D^2$. Hence, $D^2 = -12$. It is easy to derive $k = 0$ and

$$\widetilde{\mathcal{Z}} = \sigma(\nu_1 - \sigma) + (\nu_1 - 1)\omega_1 + 2\overline{g} = 4 = 3t_3 + 4t_2.$$

Then $_2 = 1$. By $Y = 8 + 8 \cdot 4 + 12 = 51 = 4t_4 + 1$, which has no solution.

(6) $\overline{g} = -1$. Then $\omega_1 = 13$ and $13 = \omega_1 = -2 - D^2$. Hence, $D^2 = -15$. It is easy to derive $k = 0$ and

$$\widetilde{\mathcal{Z}} = \sigma(\nu_1 - \sigma) + (\nu_1 - 1)\omega_1 + 2\overline{g} = 5 = 3t_3 + 4t_2.$$

This has no solution.

Therefore, we obtain the next result.

Proposition 4. *If $B \geq 3$, $\nu_1 \geq 4$ and $\omega = 12$,then the type becomes $[8 * 24, 3; 4^{12}]$ or $[8 * 24, 3; 4^{12}, 3]$.*

4. ESTIMATE OF k IN TERMS OF ω

We shall prove the following estimate of k.

Proposition 5. *If $B \leq 2$, $\sigma \geq 7$ and $\nu_1 \geq 3$,then $k \leq \omega$.*

Proof.

From proposition 4, it follows that

$$\begin{aligned} 0 \leq \widetilde{\mathcal{Z}} &= \nu_1(\omega - \overline{g} - k) - \tilde{k} - \omega_1 + 2\overline{g} \\ &= -k\nu_1 + (\nu_1 - 1)\omega_1 + 2\overline{g} - \tilde{k} \\ &= -k\nu_1 + (\nu_1 - 1)(\omega - \overline{g}) + 2\overline{g} - \tilde{k} \\ &= -k\nu_1 + (\nu_1 - 1)\omega + \overline{g}(3 - \nu_1) - \tilde{k} \\ &\leq -k\nu_1 + (\nu_1 - 1)\omega + \overline{g}(3 - \nu_1). \end{aligned}$$

Thus when $\overline{g} \geq 0$, we get

$$k\nu_1 \leq (\nu_1 - 1)\omega.$$

Hence,

$$k \leq \frac{(\nu_1 - 1)\omega}{\nu_1} < \omega.$$

However, when $\overline{g} = -1$, we get

$$k\nu_1 \leq (\nu_1 - 1)\omega + \nu_1 - 3.$$

Hence,

$$k - \omega \leq 1 - \frac{3+\omega}{<}1.$$

Therefore, $k \leq \omega$. Thus , introduce an invariant i by $i = \omega - k \geq 0$.

4.1. **case when** $k = \omega$**.** Assume $i = 0$. Then $k = \omega$ and by the previous argument, $\overline{g} = -1$.

Supposing that $\widetilde{Z} > 0$, we get $\widetilde{Z} \geq \nu_1 - 1$. Hence,

$$\begin{aligned} \nu_1 - 1 \leq \widetilde{\mathcal{Z}} &= -k\nu_1 + (\nu_1 - 1)k + \overline{g}(3 - \nu_1) - \tilde{k} \\ &= -k + \overline{g}(3 - \nu_1) - \tilde{k}. \end{aligned}$$

Thus $\overline{g} = -1$ and so

$$\nu_1 - 1 \leq -k + \nu_1 - 3 - \tilde{k}.$$

This is a contradiction. Therefore, $\widetilde{Z} = 0$.

4.2. **a formula for** i**.** In general, in the case when $i \geq 0, \overline{g} = -1$ and $\widetilde{Z} = 0$, we obtain the following formulae from the fundamental equalities (3):

- $\omega_1 = i + 1 + k$,
- $(r-8)\nu_1 = k + \omega_1 = 2k + i + 1$,
- $(r-8)\nu_1^2 = 2k\nu_1 + \omega_1 + \tilde{k} + 2 = 2k\nu_1 + \tilde{k} + i + k + 3$.

Then $r \geq 9$ and

$$\nu_1 = \frac{2k + i + 1}{r - 8}. \tag{9}$$

From

$$(r-8)\nu_1^2 = (2k + i + 1)\nu_1 = 2k\nu_1 + \tilde{k} + i + k + 3,$$

it follows that

$$(i+1)\nu_1 = \tilde{k} + i + k + 3,$$

and

$$(i+1)\frac{2k + i + 1}{r - 8} = \tilde{k} + i + k + 3.$$

Hence,

$$(i+1)(2k + i + 1) = (r - 8)(\tilde{k} + i + k + 3). \tag{10}$$

Furthermore, we obtain

$$k(2i + 10 - r) + (i+1)^2 = (r-8)(\tilde{k} + i + 3). \tag{11}$$

4.3. **case in which** $i = 0$**.** Suppose that $i = 0$. From the formula (11), it follows that

$$k(10 - r) + 1 = \tilde{k} + 3.$$

Hence, $r = 9$ and $k + 1 = \tilde{k} + 3$; $k = \tilde{k} + 2$.

Therefore, from $k = \tilde{k} + 2 = p(k - 2p) + 2$, it follows that either 1)$p = 1$ or 2) $k = 2p + 2, p \neq 1$.

In the case when $p = 1$, we have $\nu_1 = 2k + 1$ and $k = w + 2u$, where $w = 4 - \delta_{1B}$.

If $B = 1$ then $k = 3 + 2u$ and $\sigma = 2\nu_1 + p = 4k + 3$; $e = \sigma + \nu_1 + u = 6k + u + 4$.

Thus $\widetilde{B} = 2e - \sigma = 9k + 2$. The type becomes $[(4k + 3) * (6k + u + 4), 1; (2k + 1)^9]$, where $k = 3 + 2u$.

Conversely, if the minimal pair (S, D) has this type, then $g = \frac{(\sigma-1)(\widetilde{B}-2)}{2} - 9(2k+1)k = 0$ and $D^2 = \sigma\widetilde{B} - 9(2k+1)^2 = -k-3$. Thus $\omega = -3-(-k-3) = k$.

If $B = 0$ then $k = 4+2u$ and $\nu_1 = 2k+1 = 9+4u, \sigma = 2\nu_1 + p = 4k+3 = 19 + 8u$; $e = \sigma + \nu_1 + u = 19 + 9u$. The type becomes $[(19 + 8u) * (19 + 9u), 1; (9 + 4u)^9]$.

Conversely, if the minimal pair (S, D) has this type, then $g = 0$ and $\omega = 4 + 2u = k$.

In the case when $k = 2p + 2, p \neq 1$, we have either $p = 0$ or $p > 0$.

If $p = 0$ then $u = 1$ and $k = 2$. Thus $\nu_1 = 2k + 1 = 5, \sigma = 10, B = 0, 2$.

If $B = 0$ then the type becomes $[10 * 11; 5^9]$.

If $p > 1$ then $k = 2p + 2 = wp + 2u$, from which it follows that $p = 2, u = 0, w = 3, k = 6$ and $B = 1$. Moreover, $\nu_1 = 2k + 1 = 13$ and $\sigma = 28$ and $e = 41$. The type becomes $[28 * 41; 13^9]$.

Conversely, if the minimal pair (S, D) has this type then $g = 0$ and $D^2 = -9$ and $\omega = 6 = k$.

4.4. **case when** $k = \omega - 1$**.**

5. Case in which $\lambda \geq 1$

First we shall prove the inequality (2), from which the inequality (1) will be derived later.

6. Proof of the inequality (2) in the case when $k > 0$

Suppose that $\widetilde{\mathcal{Z}} = \nu_1 Y - X > 0$. Then

$$\widetilde{\mathcal{Z}} \geq \nu_r(\nu_1 - \nu_r) \geq \nu_1 - 1.$$

Recalling that

$$\widetilde{\mathcal{Z}} = \nu_1 Y - X = -\nu_1\lambda - \tilde{k} - \omega_1 + 2\overline{g}$$

we obtain

$$\nu_1 - 1 \le -\nu_1\lambda - \tilde{k} - \omega_1 + 2\overline{g},$$

where $\lambda = k - \omega_1$.

Thus,

$$\nu_1 + \lambda\nu_1 - 1 \le -\tilde{k} - \omega_1 + 2\overline{g}.$$

Since $1 - \lambda \le 0$ and $p - \tilde{k} \le 0$, it follows that

.

$$\begin{aligned}\sigma = 2\nu_1 + p \le &(1-\lambda)\nu_1 + 1 + p - \tilde{k} - \omega_1 + 2\overline{g} \\ &1 - \omega_1 + 2\overline{g}.\end{aligned}$$

Therefore,

$$\omega_1^2 + \omega_1 + 2 + 2\overline{g} - \sigma \ge (\omega_1 + 1)^2. \qquad (12)$$

Hence,

$$\omega_1^2 + \omega_1 + 2 + 2\overline{g} \ge \sigma.$$

7. CASE IN WHICH $\widetilde{\mathcal{Z}} = 0$

Suppose that $\nu_1 Y - X = \widetilde{\mathcal{Z}} = 0$. Then $\nu_1 = \cdots = \nu_r$ and hence,$X = r\nu_1^2, Y = r\nu_1$. Thus

- $(r-8)\nu_1^2 = 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $(r-8)\nu_1 = k + \omega_1$.

Recall that $\lambda = k - \omega_1 \ge 1$.

7.1. case in which $k > 0$.

(1) Suppose that $r \ge 9$. Then $\nu_1 \le (r-8)\nu_1 = k + \omega_1$. Thus,

$$\nu_1 \le k + \omega_1.$$

Hence,

$$\sigma = 2\nu_1 + p \le 2k + 2\omega_1 + p.$$

Since

$$2\lambda \le \nu_1\lambda \le -\tilde{k} - \omega_1 + 2\overline{g}, \qquad (13)$$

it follows that

$$\begin{aligned}\omega_1^2+\omega_1+2+2\overline{g}-\sigma \\ &\geq \omega_1^2+\omega_1+2+2\overline{g}-(2k+2\omega_1+p) \\ &\geq \omega_1^2+\omega_1+2+(2\lambda+\tilde{k}+\omega_1)-(2k+2\omega_1+p) \\ &\geq \omega_1^2+\omega_1+2+(2k-\omega_1+\tilde{k})-(2k+2\omega_1+p) \\ &= \omega_1^2-2\omega_1+1+1+\tilde{k}-p \\ &\geq 1.\end{aligned}$$

Therefore,if $\nu_1 \geq 2$, then

$$\omega_1^2+\omega_1+2+2\overline{g}-\sigma>0.$$

(2) Suppose that $r=8$. Then

- $0=(r-8)\nu_1^2=2k\nu_1+\tilde{k}+\omega_1-2\overline{g}$,
- $0=(r-8)\nu_1=k+\omega_1$.

Hence, $\omega_1=-k\leq -2$. Furthermore, $\lambda=k-\omega_1=2k$ and

$$2\nu_1=\frac{-\tilde{k}-\omega_1+2\overline{g}}{k};$$

thus,

$$\sigma=2\nu_1+p=\frac{-\tilde{k}-\omega_1+2\overline{g}}{k}+p.$$

Moreover, from $0=2k\nu_1+\tilde{k}+\omega_1-2\overline{g}$, it follows that

$$\begin{aligned}2\overline{g} &= 2k\nu_1+\tilde{k}+\omega_1 \\ &= 2k\nu_1+\tilde{k}-k \\ &= k(2\nu_1-1)+\tilde{k} \\ &> 3k.\end{aligned}$$

Since

$$\frac{k}{2}-\frac{2p^2}{k}=\frac{1}{2k}(k^2-4p^2)\geq\frac{5p^2}{2k}>0$$

we obtain

$$\begin{aligned}\omega_1^2+\omega_1+2+2\overline{g}-\sigma &= \omega_1^2+\omega_1+2+2\overline{g}-(\frac{-\tilde{k}-\omega_1+2\overline{g}}{k}+p)\\ &= k(k-1)+1+2\overline{g}(1-\frac{1}{k})-\frac{2p^2}{k}\\ &> k(k-1)+1+3k(1-\frac{1}{k})-\frac{2p^2}{k}\\ &\geq k^2+\frac{k}{2}-\frac{2p^2}{k}\\ &> 0.\end{aligned}$$

Thus

$$\omega_1^2+\omega_1+2+2\overline{g}-\sigma>0.$$

(3) Suppose that $r \leq 7$. Then letting s be $8-r>0$, we get

- $s\nu_1^2=-2k\nu_1-\tilde{k}-\omega_1+2\overline{g}$,
- $s\nu_1=-k-\omega_1$.

Since $\nu_1 \leq s\nu_1=-k-\omega_1$, it follows that

$$\sigma \leq -2k-2\omega_1+p.$$

Moreover,

$$\begin{aligned}\omega_1^2+\omega_1+2+2\overline{g}-\sigma &\geq \omega_1^2+3\omega_1+2+2\overline{g}+2k-p\\ &\geq \omega_1(3+\omega_1)+2+2\overline{g}+(2w-1)p+4u.\end{aligned}$$

The function defined by

$$F(x)=x(3+x)+2+2\overline{g}+(2w-1)p+4u \tag{14}$$

has minimal values at $x=-1$ or -2. By

$$F(-1)=F(-2)=-2+2+2\overline{g}+(2w-1)p+4u \geq (2w-1)p+4u-2$$

$F(x)>0$ if $k>0$.

7.2. **case in which $k=0$.**

Then

- $(r-8)\nu_1^2=\omega_1-2\overline{g}$,
- $(r-8)\nu_1=\omega_1$.

Recall that $\lambda=-\omega_1 \geq 1$. Then $\overline{g}-\omega=-\omega_1 \geq 1$ and so $\overline{g}>0, r<8$. Letting $s=8-r$, we get

- $s\nu_1^2=-\omega_1+2\overline{g}$,
- $s\nu_1=-\omega_1$.

Then $\sigma = 2\nu_1 = \frac{-2\omega_1}{s}$ and so

$$\begin{aligned}\omega_1^2 + \omega_1 + 2 + 2\overline{g} - \sigma &= \omega_1^2 + \omega_1 + 2 + 2\overline{g} + \frac{2\omega_1}{s} \\ &= (\omega_1 + \frac{s+2}{2s})^2 + 2 + 2\overline{g} - (\frac{s+2}{2s})^2\end{aligned}$$

which is positive.

If $s = 1$ then

$$2 + 2\overline{g} - (\frac{s+2}{2s})^2 = 2\overline{g} - \frac{1}{4} \geq 1.$$

If $s \geq 2$ then $\frac{s+2}{2s} < 1$. Hence,

$$\omega_1^2 + \omega_1 + 2 + 2\overline{g} > \sigma.$$

8. CASE IN WHICH $\lambda \leq 0$

Suppose that $\lambda = k - \omega + \overline{g} \leq 0$. In other words, $\omega - \overline{g} \geq k = wp + 2u$.

First we note the following lemma, which is a bit sharper result than the lemma proved by Matsuda.

8.1. Lemma due to Tanaka and Matsuda.

Lemma 2 (Tanaka and Matsuda). *Let $m, \mu_1, \cdots, \mu_r$ be integers such that $m \geq \mu_1 \geq \cdots \geq \mu_r \geq 2$ and that $\sum_{j=1}^r \mu_j = sm + \beta$ for some integers $s \geq 0$ and $\beta > 0$.*

Putting $X = \sum_{j=1}^r \mu_j^2, Y = \sum_{j=1}^r \mu_j$, we obtain $Y = sm + \beta$ and $V = sm^2 + \beta^2 - X$ which satisfy that

- $V \geq 0$.
- *If $V = 0$ then either 1) $m = \mu_1 = \cdots = \mu_{r-1} > \mu_r$ and $s = r-1, \beta = \mu_r$ or 2) $m = \mu_1 = \cdots = \mu_r$ and $s = r-1, \beta = m$.*
- *If $V > 0$ then $V \geq 2$.*
- *If $V = 2$ then $m = \mu_1 = \cdots = \mu_{r-2} > \mu_{r-1} = \mu_r = m-1, s = r-1, \beta = m-2$.*
- *If $V > 2$ then $V \geq 4$.*
- *If $V = 4$ then $m = \mu_1 = \cdots = \mu_{r-2} > \mu_{r-1} = m-1, \mu_r = m-2, s = r-1, \beta = m-4$.*

Proof.

(1) Assume that $\beta \geq m$. Dividing β by m, we have q, r_0 such that $\beta = qm + r_0$, $0 \leq r_0 < m$ and $q \geq 1$. Then $Y = sm + \beta = (s+q)m + r_0$ and let $s' = s + q$. Thus

$$V' = s'm^2 + r_0^2 - X, V = sm^2 + \beta^2 - X.$$

Hence,

$$V - V' = sm^2 + \beta^2 - (s'm^2 + r_0^2) = -qm^2 + (qm + r_0)^2 - r_0^2 = qm((q-1)m + 2r_0) \geq 0.$$

If $V = V'$ then $q = 1, r_0 = 0$. In this case, $\beta = m$ and $Y = sm + m = (s+1)m$.

Otherwise, $V \geq V' + 2m \geq V' + 4$. If $V = V' + 4$ then $m = 2$.
Thus we assume $\beta < m$.

(2) If $\mu_1 = m$, then replace m by $m-1$ and r by $r-1$,respectively. After such replacement, V is invariant. Hence, we may assume that $\mu_1 < m$ and we shall prove the lemma by induction on r.

(3) If $r = 1$ then $s = 0$ and $\mu_1 = \beta$; thus $V = 0$.

(4) When $r > 1$, (i) we suppose that $\mu_1 + \mu_2 < m$. Then letting $\mu_1' = \mu_1 + \mu_2$, we define X' and Y' as follows:

- $X' = {\mu_1'}^2 + \sum_{j=3}^{r} {\mu_j}^2$,
- $Y' = \mu_1' + \sum_{j=3}^{r} \mu_j$.

Since $Y = Y' = sm + \beta$, from induciton hypothesis,it follows that

$$V' = sm^2 + \beta^2 - X' \geq 0.$$

But $V = sm^2 + \beta^2 - X$ satisfies that

$$V - V' = X' - X = (\mu_1 + \mu_2)^2 - (\mu_1^2 + \mu_2^2) = 2\mu_1\mu_2 \geq 8.$$

(ii) Assume $\mu_1 + \mu_2 > m + 1$. Then $2m - 2 \geq \mu_1 + \mu_2$ and putting $\mu_1' = m, m - 2 \geq \mu_2' = \mu_1 + \mu_2 - m \geq 2$, we define X' and Y' as follows:

- $X' = {\mu_2'}^2 + \sum_{j=3}^{r} \mu_j^2$,
- $Y' = \mu_2' + \sum_{j=3}^{r} \mu_j$.

Then $Y' = (s-1)m + \beta$ and $X = {\mu_1}^2 + \mu_2^2 - \mu_2'^2 + X'$. By induction hypothesis, $V' = (s-1)m^2 + \beta^2 - X' \geq 0$ and

$$V = sm^2 + \beta^2 - X, \quad V' = (s-1)m^2 + \beta^2 - X'.$$

Thus

$$V - V' = m^2 + X' - X = (\mu_1 + \mu_2 - m)^2 + m^2 - (\mu_1^2 + \mu_2^2).$$

Note the following lemma.

Lemma 3. *Let a, b, m be nonnegative integers satisfying that*

$$2 \leq a \leq m-1, 2 \leq b \leq m-1, \textit{and } m+2 \leq a+b.$$

Then

$$m^2 + (a+b-m)^2 \geq a^2 + b^2 + 2.$$

If the equality holds, then $a = m-1, b = m-1$.

Proof. Define a function $F(x) = m^2 + (x+b-m)^2 - (x^2 + b^2 + 2) = m^2 + 2(b-m)x + (b-m)^2 - b^2 - 2$, which is a liner function. Since $b - m \leq -1$ and $x \leq m-1$, it suffices to show that $F(m-1) \geq 0$. However, $F(m-1) = 2(m-b-1) \geq 0$. Furthermore, if $F(x) = 0$ then $x = m-1$ and $b = m-1$. Q.E.D.

Applying the lemma, we see that $V - V' \geq 2$. And if $V - V' = 2$ then $\mu_1 = \mu_2 = m - 1$.

(iii) Assume $\mu_1 + \mu_2 = m$. Then Y'' and X'' defined below

- $X'' = \sum_{j=3}^{r} \mu_j^2 = s'm + \beta'$,
- $Y'' = \sum_{j=3}^{r} \mu_j$,

satisfy that $Y = m + Y''$ and $X = \mu_1^2 + \mu_2^2 + X''$. Moreover, $Y = s'm + \beta' + m = sm + \beta$.

Then we have $s = s' + 1, \beta' = \beta$. Futher, since $m^2 - (\mu_1^2 + \mu_2^2) = 2\mu_1\mu_2 > 0$, it follows that

$$\begin{aligned} V &= sm^2 - X \\ &= s'm^2 + m^2 - (\mu_1^2 + \mu_2^2 + X'') \\ &= V'' + m^2 - (\mu_1^2 + \mu_2^2) \\ &= V'' + 2\mu_1\mu_2 \\ &\geq V'' + 8. \end{aligned}$$

(iv) Assume $\mu_1 + \mu_2 = m + 1$. Then $Y = m + 1 + Y'' = s'm + \beta' + m + 1 = sm + \beta$.

If $\beta' < m - 1$ then $s = s' + 1, \beta = \beta' + 1$. Thus $Y = m(s' + 1) + \beta' + 1$ and

$$\begin{aligned} V =& m^2(s' + 1) + (\beta' + 1)^2 - X \\ =& s'm^2 + m^2 + \beta'^2 - (X'' + \mu_1^2 + \mu_2^2) + 2\beta' + 1 \\ =& V'' + m^2 + 2\beta' + 1 - \mu_1^2 - \mu_2^2 \\ \geq& V'' + (\mu_1 + \mu_2 - 1)^2 + 1 - \mu_1^2 - \mu_2^2 \\ =& V'' + 2\mu_1\mu_2 - 2\mu_1 - 2\mu_2 + 2 \\ =& V'' + 2(\mu_1 - 1)(\mu_2 - 1) \\ \geq& V'' + 2. \end{aligned}$$

If $V = V' + 2$ then $\mu_1 = \mu_2$ and $\beta' = 0$.

Moreover, if $\beta' = m - 1$ then $Y = m(s' + 2)$. Thus

$$\begin{aligned} V = m^2(s' + 2) - X &= m^2(s' + 2) - (X'' + \mu_1^2 + \mu_2^2) \\ &\geq V'' + 2m^2 - \mu_1^2 - \mu_2^2 \\ &= V'' + m^2 - \mu_1^2 + m^2 - \mu_2^2 \\ &\geq V'' + 8 \geq 0. \end{aligned}$$

Here, $m^2 - \mu_1^2 \geq (\mu_1 + 1)^2 - \mu_1^2 = 2\mu_1 + 1$. Q.E.D.

9. PROOF OF THE INEQUALITY (2) WHEN $\lambda \le 0$

9.1. case in which $k > 0$.

By applying Lemma of Tanaka and Matsuda to the following

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $Y = 8\nu_1 + k + \omega_1$,

we see that $V = 8\nu_1^2 + (k+\omega_1)^2 - X \ge 0$. Hence,

$$V = (k+\omega_1)^2 - (2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}) \ge 0.$$

Thus

$$2k\nu_1 \le (k+\omega_1)^2 - (\tilde{k} + \omega_1 - 2\overline{g}).$$

Assume $k > 0$. Then

$$\sigma = 2\nu_1 + p \le \frac{(k+\omega_1)^2 - (\tilde{k} + \omega_1 - 2\overline{g}) + kp}{k}.$$

Furthermore, we get

$$\begin{aligned}\sigma &= 2\nu_1 + p \\ &\le \frac{(k+\omega_1)^2 - (\tilde{k} + \omega_1 - 2\overline{g}) + kp}{k} \\ &= \frac{(k+\omega_1)^2 + 2p^2 - \omega_1 + 2\overline{g}}{k} \\ &= k + 2\omega_1 + \frac{{\omega_1}^2 - \omega_1 + 2p^2 + 2\overline{g}}{k}.\end{aligned}$$

9.1.1. *example.*

Example 2. *If the type of* (S, D) *is* $[2\nu_1 * 2\nu_1; \nu_1^r]$ *then*

$$g = (2\nu_1 - 1)^2 - r \times \frac{\nu_1(\nu_1 - 1)}{2}, \quad D^2 = (8-r){\nu_1}^2.$$

Hence, $\omega = \frac{(8-r)\nu_1(\nu_1-3)}{2}$.

TABLE 16. ω

r	7	6	5
ν_1			
4	2	4	6
5	5	10	15
6	9	18	27

9.2. case in which $p \geq 1$.

Suppose that $p \geq 1$. Then $k \geq 3p \geq 3$, so we have $\frac{2p^2}{k} \leq \frac{2}{9}k$. Recalling that $\omega_1 \geq k$, we get

$$\begin{aligned}\sigma &\leq (1+\frac{2}{9})k + 2\omega_1 + \frac{{\omega_1}^2 - \omega_1 + 2\overline{g}}{k} \\ &\leq (3+\frac{2}{9})\omega_1 + \frac{{\omega_1}^2 - \omega_1 + 2\overline{g}}{k} \\ &\leq (\frac{10}{3})\omega_1 + \frac{{\omega_1}^2 - \omega_1 + 2\overline{g}}{3} \\ &= \frac{{\omega_1}^2 + 9\omega_1 + 2\overline{g}}{3}.\end{aligned}$$

Therefore, we obtain

Proposition 6. *If $p \geq 1$ then*

$$\sigma \leq (1+\frac{2}{9})k + 2\omega_1 + \frac{{\omega_1}^2 - \omega_1 + 2\overline{g}}{k}.$$

In particular,

$$\sigma \leq \frac{{\omega_1}^2 + 9\omega_1 + 2\overline{g}}{3}. \tag{15}$$

However, we shall show that

$$\frac{{\omega_1}^2 + 9\omega_1 + 2\overline{g}}{3} \leq {\omega_1}^2 + \omega_1 + 2 + 2\overline{g}.$$

This is equivalent to the following

$${\omega_1}^2 + 9\omega_1 + 2\overline{g} \leq 3({\omega_1}^2 + \omega_1 + 2 + 2\overline{g}).$$

Defining a function $F(x)$ to be $x^2 - 3x + 3 + 2\overline{g}$, we see that the difference of the both sides of the above inequality is written as $2F(\omega_1)$.

By $\omega_1 \geq k \geq 3$, to verify the above inequality it suffices to show that $F(3) \geq 0$. But

$$F(3) = 3 + 2\overline{g} \geq 1.$$

Thus we have established that

$$\begin{aligned}\omega^2 + (1-2\overline{g})\omega + \overline{g}^2 + \overline{g} + 1 &= {\omega_1}^2 + \omega_1 + 2 + 2\overline{g} \\ &\geq \frac{{\omega_1}^2 + 9\omega_1 + 2\overline{g}}{3} + \frac{2F(3)}{3} \\ &\geq \sigma + \frac{6+4\overline{g}}{3}.\end{aligned}$$

Hence,

$$\omega^2 + (1-2\overline{g})\omega + \overline{g}^2 + \overline{g} + 1 \geq \sigma + 2 + \frac{4\overline{g}}{3} > \sigma.$$

9.3. case in which $p = 0, u \geq 1$.

Suppose that $p = 0$. Then $k = 2u \geq 2$, so we have by $\omega_1 \geq k = 2u$,

$$\begin{aligned}\sigma = 2\nu_1 &\leq \frac{(2u+\omega_1)^2 - \omega_1 + 2\overline{g}}{2u} \\ &\leq 2u + 2\omega_1 + \frac{{\omega_1}^2 - \omega_1 + 2\overline{g}}{2u} \\ &\leq 3\omega_1 + \frac{{\omega_1}^2 - \omega_1 + 2\overline{g}}{2}.\end{aligned}$$

Next, we shall show that

$$3\omega_1 + \frac{{\omega_1}^2 - \omega_1 + 2\overline{g}}{2} < {\omega_1}^2 + \omega_1 + 2 + 2\overline{g}.$$

This is equivalent to the following

$${\omega_1}^2 - \omega_1 + 2\overline{g} < 2{\omega_1}^2 - 4\omega_1 + 4 + 4\overline{g}.$$

But

$$\begin{aligned}&2{\omega_1}^2 - 4\omega_1 + 4 + 4\overline{g} - ({\omega_1}^2 - \omega_1 + 2\overline{g}) \\ &= {\omega_1}^2 - 3\omega_1 + 4 + 2\overline{g}.\end{aligned}$$

Since $\omega_1 \geq 2u \geq 2$, it follows that

$${\omega_1}^2 - 3\omega_1 + 4 + 2\overline{g} \geq 2 + 2\overline{g} \geq 0.$$

If the equality holds, then $\overline{g} = -1, \omega_1 = 2$. Note the following lemma:

Lemma 4. *If $p = 0$ and $\omega_1 = k$ then $2\overline{g} \geq \omega_1$.*

Proof. By the fundamental equalities :

- $X = 8\nu_1^2 + 2k\nu_1 + \omega_1 - 2\overline{g}$,
- $Y = 8\nu_1 + 2k$

we get $0 \leq \nu_1 Y - X = 2\overline{g} - \omega_1$. Q.E.D.

Then $\omega_1 = 2$ implies $k = 2$ and so $2\overline{g} \geq \omega_1 = 2$. Therefore,

$${\omega_1}^2 - 3\omega_1 + 4 + 2\overline{g} = 2 + 2\overline{g} \geq 4.$$

Thus, we have established if $\omega_1 = 2$, then

$${\omega_1}^2 + \omega_1 + 2 + 2\overline{g} \geq \sigma + 2.$$

If $\omega_1 = 3$ then

$${\omega_1}^2 - 3\omega_1 + 4 + 2\overline{g} = 4 + 2\overline{g} \geq 2.$$

If the equality holds, then $g = 0$ and $k = 2, u = 1$. By

- $X = 8\nu_1^2 + 2k\nu_1 + \omega_1 - 2\overline{g} = 8\nu_1^2 + 4\nu_1 + 5$,
- $Y = 8\nu_1 + k + \omega_1 = 8\nu_1 + 5$

we get $0 \leq \nu_1 Y - X = \nu_1 - 5$; thus, $\nu_1 \geq 5$.

From

$$10 \leq \sigma = 2\nu_1 \leq 3\omega_1 + \frac{{\omega_1}^2 - \omega_1 + 2\overline{g}}{2} = 11$$

it follows that $\nu_1 \leq 5$; hence $\nu_1 = 5$. This implies that $\nu_1 Y - X = \nu_1 - 5 = 0$. Hence, $\nu_1 = 5$. Thus the type of the pair is associated with $[10 * 11; 5^r]$. By $g = 90 - 10r = 10(9 - r) = 0$, we have $r = 9$, $\omega = 2, \sigma = 10$, and ${\omega_1}^2 + \omega_1 + 2 + 2\overline{g} = 12$ Except for this type, we obtain

$${\omega_1}^2 + \omega_1 + 2 + 2\overline{g} \geq \sigma + 4.$$

10. Proof of the inequality (2) in the case when $\lambda \leq 0$ and $k = 0$

Suppose that $k = 0$, namely, $p = u = 0$. As before, $\lambda = -\omega + \overline{g} \leq 0$.

The fundamental equalities imply

- $X = 8\nu_1^2 + \omega_1 - 2\overline{g}$,
- $Y = 8\nu_1 + \omega_1$.

Following Matsuda([9]), let t denote t_{ν_1}. Then let X' be $\sum_{\nu_j < \nu_1} {\nu_j}^2$ and $Y' = \sum_{\nu_j < \nu_1} \nu_1$. Hence, $X = X' + t{\nu_1}^2$ and $Y = Y' + t\nu_1$. Therefore,

- $X' = (8 - t)\nu_1^2 + \omega_1 - 2\overline{g}$,
- $Y' = (8 - t)\nu_1 + \omega_1$.

10.1. case in which $t \geq 8$.

If $t \geq 8$ then let s denote $8 - t$, namely $s = 8 - t \leq 0$. Thus

- $X' - s\nu_1^2 = \omega_1 - 2\overline{g}$,
- $Y' - s\nu_1 = \omega_1$.

Therefore,

$$X' - s\nu_1^2 - (Y' - s\nu_1) = -2\overline{g} \leq 2;$$

hence,

$$X' - Y' - s(\nu_1^2 - \nu_1) \leq 2.$$

By $\nu_1 \geq 4$, we get $\nu_1^2 - \nu_1 \geq 12$. Hence,$s = 0$,i.e., $t = 8$ and if $r > 8$ then $2 \geq X' - Y' \geq \nu_r(\nu_r - 1) \geq 2$. Thus $X' = 4, Y' = 2, \nu_r = 2, r = 9$ which implies that the type is associated with $[2\nu_1 * 2\nu_1; \nu_1^8, 2]$. Otherwise, $r = 8$.

Consequently, we have the following contradictory result:

If $g = 1$ then $r = 8$ and the type is associated with $[2\nu_1 * 2\nu_1; \nu_1^8]$. But then $\kappa[D] < 2$, which contradicts the hypothesis.

If $g = 0$ then $r = 9$ and the type is associated with $[2\nu_1 * 2\nu_1; \nu_1^8, 2]$. But again $\kappa[D] < 2$, which contradicts the hypothesis.

10.2. **case in which** $t < 8$.

Thus $t < 8$. Hence, $s = 8 - t > 0$ and

- $X' = s\nu_1^2 + \omega_1 - 2\overline{g}$,
- $Y' = s\nu_1 + \omega_1$.

Defining $\varepsilon(t)$ to be $\sum_{j=2}^{\nu_1-1} t_j$, we get

$$s(\nu_1 - 1) < s\nu_1 + \omega_1 = Y' = \sum_{j=2}^{\nu_1-1} jt_j \leq (\nu_1 - 1)\varepsilon(t),$$

and so

$$s < \varepsilon(t).$$

By the way, from

$$(\nu_1 - 1)\omega_1 + 2\overline{g} = \widetilde{\mathcal{Z}} \geq (\nu_1 - 1)\varepsilon(t).$$

it follows that

$$\varepsilon(t) \leq \omega_1 + \frac{2\overline{g}}{\nu_1 - 1}.$$

Thus,

Proposition 7.

$$s \leq \omega_1 + \frac{2\overline{g}}{\nu_1 - 1} - 1. \tag{16}$$

10.3. **quadratic estimate.** Following Matsuda([9]), applying the lemma for $m = \nu_1 - 1$, since $Y' = s(\nu_1 - 1) + s + \omega_1$, we have

$$V = s(\nu_1 - 1)^2 + (s + \omega_1)^2 - X' \geq 0.$$

Hence,

$$s\nu_1^2 + \omega_1 - 2\overline{g} \geq s(\nu_1 - 1)^2 + (s + \omega_1)^2.$$

Then, we get

$$s\nu_1^2 + \omega_1 - 2\overline{g} \leq s(\nu_1 - 1)^2 + (s + \omega_1)^2,$$

and then

$$\sigma = 2\nu_1 \leq \frac{s + (s + \omega_1)^2 - (\omega_1 - 2\overline{g})}{s}.$$

Thus,

$$\sigma \leq s + 2\omega_1 + 1 + \frac{{\omega_1}^2 - \omega_1 + 2\overline{g}}{s}.$$

By Lemma of Matsuda and Tanaka,if $V = 0$ in other words,the equality holds, then we have two cases:

(1) $s + \omega_1 = \nu_1 - 1$ and so $Y' = (s + 1)(\nu_1 - 1), r - t = s + 1$. Hence, $r = 9$ and the type is associated with $[2\nu_1 * 2\nu_1; \nu_1^t, (\nu_1 - 1)^{9-t}]$.

(2) $t_{\nu_1 - 1} = r - t - 1 = s - 1 = 8 - t - 1$. Then $r = 9$ and the type is associated with $[2\nu_1 * 2\nu_1; \nu_1^t, (\nu_1 - 1)^{8-t}, \nu_r]$. By computation, $g = \nu_1 - \nu_r(\nu_r - 1)/2$.

10.4. **case (1).** In case (1), we have $s + \omega_1 = \nu_1 - 1$ and from

$$(s+1)(\nu_1 - 1)^2 = X' = s\nu_1^2 + \omega_1 - 2\overline{g}$$

it follows that

$$2\overline{g} = (1 - \nu_1)(\nu_1 - 2(9 - t)) = (\nu_1 - 1)(2s + 2 - \nu_1). \tag{17}$$

This implies that

$$\begin{aligned} 2\omega &= 2\nu_1 - 2 + 2\overline{g} - 2s \\ &= 2\nu_1 - 2 + \nu_1(2s + 3 - \nu_1) - 2s - 2 - 2s \\ &= \nu_1(2s + 5 - \nu_1) - 4s - 4 \\ &= \nu_1(21 - 2t - \nu_1) - 36 + 4t. \end{aligned}$$

Thus we obtain

$$\omega = \frac{\nu_1(21 - \nu_1)}{2} - 18 - t(\nu_1 - 2). \tag{18}$$

We distinguish the various cases according to the value of $\overline{g}$.

(i) $\overline{g} = -1$,
(ii) $\overline{g} = 0$,
(iii) $\overline{g} > 0$.

10.5. **case (i).**
(i) $\overline{g} = -1$.
Then

$$2 = -2\overline{g} = (\nu_1 - 1)(\nu_1 - 2(9 - t)).$$

Thus, $\nu_1 = 2$ or 3; in other words, $\sigma = 4$ or 6.
This contradicts the hypothesis saying $\sigma \geq 7$.

10.6. **case (ii).**
(ii) $\overline{g} = 0$. Then $2s + 2 = \nu_1$ and from $s + \omega = s + \omega_1 = \nu_1 - 1 = 2s + 1$, it follows that

$$\omega = s + 1 = 9 - t, \sigma = 2\nu_1 = 4s + 4.$$

The type becomes $[(4s+4) * (4s+4); (2s+2)^{8-s}, (2s+1)^{s+1}]$.

Then

$$0 = -2\overline{g} = (\nu_1 - 1)(\nu_1 - 2(9 - t)).$$

Hence, $\nu_1 = 2s + 2 = 2(9 - t)$; thus $\sigma = 2\nu_1 = 4s + 4$.
But, since $\overline{g} = 0$, it follows that $\omega = \omega_1 = \nu_1 - 1 - s = s + 1$.

TABLE 17. $\overline{g} = 0$

t	s	ω	ν_1	σ	type
7	1	2	4	8	$[8*8;4^7,3^2]$
6	2	3	6	12	$[12*12;6^6,5^3]$
5	3	4	8	16	$[16*16;8^5,7^4]$
4	4	5	10	20	$[18*18;9^4,8^5]$
3	5	6	12	24	$[20*20;10^3,9^6]$
2	6	7	14	28	$[22*22;11^2,10^7]$
1	7	8	16	32	$[24*24;12^1,11^8]$

10.7. **case (iii).**
(iii) $\overline{g} > 0$.
By $2\overline{g} = (\nu_1 - 1)(2s + 2 - \nu_1) > 0$, we get the bound of ν_1;indeed,

$$4 \leq \nu_1 \leq 2s + 1. \tag{19}$$

Hence, $s \geq 2$.

TABLE 18. $s = 2$

ν_1	$\nu_1 - 1$	$6 - \nu_1$	$\overline{g}$	ω	type
4	3	2	3	4	$[8*8;4^6,3^3]$
5	4	1	2	4	$[10*10;5^6,4^3]$

TABLE 19. $s = 3$

ν_1	$\nu_1 - 1$	$8 - \nu_1$	$\overline{g}$	ω	type
4	3	4	6	6	$[8*8;4^5,3^4]$
5	4	3	6	7	$[10*10;5^5,4^4]$
6	5	2	5	7	$[12*12;6^5,5^4]$
7	6	1	3	6	$[14*14;7^5,6^4]$

TABLE 20. $s = 4$

ν_1	$\nu_1 - 1$	$10 - \nu_1$	$\overline{g}$	ω	type
4	3	6	9	8	$[8*8;4^4,3^5]$
5	4	5	10	10	$[10*10;5^4,4^5]$
6	5	4	10	11	$[12*12;6^4,5^5]$
7	6	3	9	11	$[14*14;7^4,6^5]$
8	7	2	7	10	$[16*16;8^4,7^5]$
9	8	1	4	8	$[18*18;9^4,8^5]$

TABLE 21. $s = 5$

ν_1	$\nu_1 - 1$	$12 - \nu_1$	$\overline{g}$	ω	type
4	3	8	12	10	$[8 * 8; 4^3, 3^6]$
5	4	7	14	13	$[10 * 10; 5^3, 4^6]$
6	5	6	15	15	$[12 * 12; 6^3, 5^6]$
7	6	5	15	16	$[14 * 14; 7^3, 6^6]$
8	7	4	14	16	$[16 * 16; 8^3, 8^6]$
9	8	3	12	15	$[18 * 18; 9^3, 8^6]$
10	9	2	9	13	$[20 * 20; 10^3, 9^6]$
11	10	1	5	10	$[22 * 22; 11^3, 10^6]$

TABLE 22. $s = 6$

ν_1	$\nu_1 - 1$	$14 - \nu_1$	$\overline{g}$	ω	type
4	3	10	15	12	$[8 * 8; 4^2, 3^7]$
5	4	9	18	16	$[10 * 10; 5^2, 4^7]$
6	5	8	20	19	$[12 * 12; 6^2, 5^7]$
7	6	7	21	21	$[14 * 14; 7^2, 6^7]$
8	7	6	21	22	$[16 * 16; 8^2, 7^7]$
9	8	5	20	22	$[18 * 18; 9^2, 8^7]$
10	9	4	18	21	$[20 * 20; 10^2, 9^7]$
11	10	3	15	19	$[22 * 22; 11^2, 9^7]$
12	11	2	11	16	$[24 * 24; 12^2, 11^7]$
13	12	1	6	12	$[26 * 26; 13^2, 12^7]$

TABLE 23. $s = 7$

ν_1	$\nu_1 - 1$	$16 - \nu_1$	$\overline{g}$	ω	type
4	3	12	18	14	$[8 * 8; 4, 3^8]$
5	4	11	22	19	$[10 * 10; 5, 4^8]$
6	5	10	25	23	$[12 * 12; 6, 5^8]$
7	6	9	27	26	$[14 * 14; 7, 6^8]$
8	7	8	28	28	$[16 * 16; 8, 7^8]$
9	8	7	28	29	$[18 * 18; 9, 8^8]$
10	9	6	27	29	$[20 * 20; 10, 9^8]$
11	10	5	25	28	$[22 * 22; 11, 10^8]$
12	11	4	22	26	$[24 * 24; 12, 11^8]$
13	12	3	18	23	$[26 * 26; 13, 12^8]$
14	13	2	13	19	$[28 * 28; 14, 13^8]$
15	14	1	7	14	$[30 * 30; 15, 14^8]$

10.8. **case (2).** Second, we study case (2):

(2) $t_{\nu_1 - 1} = r - t - 1 = s - 1 = 8 - t - 1$. Then $r = 9$ and the type is associated with $[2\nu_1 * 2\nu_1; \nu_1^t, (\nu_1 - 1)^{8-t}, \nu_r]$. By computation, $g = \nu_1 - \nu_r(\nu_r - 1)/2$.

10.9. **case in which $s = 1$.**
Assume $s = 1$. Then $t = 7$ and $g = \nu_1 - \nu_r(\nu_r - 1)/2$; thus,

$$\sigma = 2\nu_1 \leq 1 + (1 + \omega_1)^2 - (\omega_1 - 2\overline{g}).$$

If $\overline{g} = -1$ then we obtain

$$\sigma = 2\nu_1 \leq (\omega + 2)(\omega + 1).$$

Here, the equality holds if and only if $\sigma = (\omega + 1)(\omega + 2)$.

Moreover, $\nu_1 = \frac{\nu_r(\nu_r - 1)}{2}$;hence, the type is associated with $[2\nu_1 * 2\nu_1; \nu_1^7, \nu_1 - 1, \nu_r]$. Then $\omega = \nu_r - 2$.

(1) If $\nu_r = 4$, then $\nu_1 = 6$,the type is associated with $[12 * 12; 6^7, 5, 4]$.
(2) If $\nu_r = 5$, then $\nu_1 = 10$,the type is associated with $[20 * 20; 10^7, 9, 5]$,
(3) If $\nu_r = 6$, then $\nu_1 = 15$,the type is associated with $[30 * 30; 15^7, 14, 6]$.

If $\overline{g} = 0$, then $g = 1$ and $\nu_1 = \frac{\nu_r(\nu_r - 1)}{2} + 1$

$$\sigma = 2\nu_1 \leq \omega^2 + \omega + 2.$$

If $\overline{g} = 1$ then we obtain

$$\sigma = 2\nu_1 \leq \omega^2 + \omega - 2.$$

If $\overline{g} = 2$ then we obtain

$$\sigma = 2\nu_1 \leq \omega^2 - \omega + 4.$$

10.10. **case in which $s = 2$.** If $s = 2$ and $\overline{g} = -1$ then $t = 6$ and we obtain

$$\sigma = 2\nu_1 \leq \frac{(\omega + 2)(\omega + 3) + 2}{2} = \frac{\omega^2 + 5\omega + 8}{2}.$$

10.11. **case in which $s \geq 2$.** Here we shall prove the inequality (2). First note the following

$$\begin{aligned} \sigma &= 2\nu_1 \\ &\leq \frac{s + (s + \omega_1)^2 - (\omega_1 - 2\overline{g})}{s} \\ &= 1 + s + 2\omega_1 + \frac{\omega_1^2 - \omega_1 + 2\overline{g}}{s}. \end{aligned}$$

That is

$$\sigma \leq 1 + s + 2\omega_1 + \frac{\omega_1^2 - \omega_1 + 2\overline{g}}{s}. \tag{20}$$

Subtracting $1+s+2\omega_1+\frac{\omega_1^2-\omega_1+2\overline{g}}{s}$ from

$$\omega_1^2+\omega_1+2+2\overline{g}$$

we have

$$(s-1)(\omega_1^2-\omega_1+2\overline{g}-s).$$

Therefore, if

$$\omega_1^2-\omega_1+2\overline{g}\geq s$$

then

$$\omega_1^2+\omega_1+2+2\overline{g}\geq 1+s+2\omega_1+\frac{\omega_1^2-\omega_1+2\overline{g}}{2}\geq\sigma.$$

Thus, we obtain the inequality (2).

Therefore, assuming that

$$\omega_1^2-\omega_1+2\overline{g}<s, \tag{21}$$

we shall derive a contradiction, referring to the inequality in Proposition 7.

As a matter of fact,

$$\omega_1^2-\omega_1+2\overline{g}<\omega_1+\frac{2\overline{g}}{\nu_1-1}-1.$$

Hence,

$$\omega_1^2-2\omega_1+2\overline{g}(1+\frac{1}{\nu_1-1})+1<0.$$

Thus, $\overline{g}=-1$. Then $\omega_1=\omega+1\geq 3$.

$$0>\omega_1^2-2\omega_1+2\overline{g}(1+\frac{1}{\nu_1-1})+1=(\omega_1+1)^2-2(1+\frac{1}{\nu_1-1})>0.$$

This is a contradiction. Thus, the proof of the inequality (2) is complete.

11. PROOF OF THE INEQUALITY (1)

We shall derive the inequality (1) from the inequality (2).

If $g = 0$ then the inequality (2) turns out to be the inequality (1). Hence, we assume $g \geq 1$. From

$$\begin{aligned}\omega^2 + 3\omega + 2 - (\omega_1^2 + \omega_1 + 2 + 2\overline{g}) &= (\omega - \omega_1)(\omega + \omega_1 + 1) - 2\overline{g} + 2\omega \\ &= g(\omega + \omega_1) \\ &= g(2\omega - \overline{g}),\end{aligned}$$

it follows that when $2\omega - \overline{g} \geq 0$, the inequality (1) is derived.

Hence, we assume

$$2\omega - \overline{g} < 0. \tag{22}$$

However,

$$4\omega - 2\overline{g} = 4(3\overline{g} - D^2) - 2\overline{g} = 2(5\overline{g} - 2D^2).$$

By Hartshorne's lemma, we have either (1) $|2D + \sigma K_S| \neq \emptyset$ or (2) $B = 1$ and $|3D + eK_S| \neq \emptyset$.

11.1. case (1).

In case (1), we have $(2D + \sigma K_S) \cdot D = 2\sigma\overline{g} - (\sigma - 2)D^2 \geq 0$ and then

$$5\overline{g} - 2D^2 \geq \frac{\sigma - 10}{2\sigma} D^2.$$

(i) If $D^2 \leq 0$ then $5\overline{g} - 2D^2 \geq 5\overline{g} \geq 0$.

(ii) If $D^2 > 0$ and $\sigma - 10 \geq 0$ then

$$5\overline{g} - 2D^2 \geq \frac{\sigma - 10}{2\sigma} D^2 \geq 0.$$

Thus we may assume $\sigma < 10$. Thus in order to prove

$$\sigma \leq \omega^2 + 3\omega + 2$$

it suffices to assume that $\omega = 1$. Recalling that $7 \leq \sigma < 10$, we have $\omega_1 = 1 - \overline{g}$ and

$$\begin{aligned}\widetilde{\mathcal{Z}} &= -k\nu_1 + (\nu_1 - 1)\omega_1 + 2\overline{g} \\ &= -k\nu_1 + (\nu_1 - 1)(1 - \overline{g}) + 2\overline{g} \\ &= \nu_1(1 - k - \overline{g}) + 3\overline{g} - 1.\end{aligned}$$

If $\nu_1 = 3$ then $p = \sigma - 6 > 0, k > 0$ and $\widetilde{\mathcal{Z}} = 3 - 3k - 1 < 0$, a contradiction.

If $\nu_1 = 4$ then $p = \sigma - 8 \leq 1$. But from (22),it follows that $\overline{g} > 2\omega = 2$. Therefore,

$$\widetilde{\mathcal{Z}} = 4 - 4k - \overline{g} - 1 \leq 0.$$

Hence, $k = 0, p = 0$ and $\sigma = 8$. Moreover, $\widetilde{\mathcal{Z}} = 4 - \overline{g} - 1 = 0$.

By the way, from $\widetilde{\mathcal{Z}} = 0$, it follows that $g = 4$ and $g = 49 - 6t_4 = 4$, which has no solution.

11.2. case (2).

In case (2), we have $(3D + eK_S) \cdot D = 2e\overline{g} - (e-3)D^2 \geq u + \nu_1 > 0$ and then

$$\omega \geq \frac{2 + (e-9)\overline{g}}{e-3}.$$

Moreover,

$$2\omega - \overline{g} \geq \frac{e-11}{e-3}\overline{g}.$$

Hence, we may assume that $e - 11 < 0$. However, $10 \geq e = \sigma + u + \nu_1 \geq \sigma + 3$. Thus, $\nu_1 = 3, \sigma = 7, k > 0$. Finally,

$$\widetilde{\mathcal{Z}} = -3k + 2(1 - \overline{g}) + 2\overline{g} = 2 - 3k < 0.$$

This is a contradiction. Q.E.D.

12. An inequality for curves with $g > 0$

Namely, we shall verify the following

Theorem 6. *If $\sigma \geq 7$ and $g \geq 1$ then*

$$\sigma \leq \omega^2 + \omega + 2 \tag{23}$$

except for the type $[7 * 9, 1; 1]$. In this case, $\sigma = 7$ and $\omega = 1$; the right hand side is 4.

The right hand side of the inequality is obtained from that of the next inequality after putting $g = 1$.

$$\sigma \leq \omega_1^2 + \omega_1 + 2 + 2\overline{g}. \tag{24}$$

Proof. May assume that $g \geq 2$. By

$$\begin{aligned} \omega^2 + \omega + 2 - (\omega_1^2 + \omega_1 + 2 + 2\overline{g}) &= (\omega - \omega_1)(\omega + \omega_1 + 1) - 2\overline{g} \\ &= \overline{g}(\omega + \omega_1 - 1) \\ &= \overline{g}(2\omega - g), \end{aligned}$$

if $2\omega \geq g$ then

$$\omega^2 + \omega + 2 \geq \omega_1^2 + \omega_1 + 2 + 2\overline{g} \geq \sigma.$$

Note that $2\omega - g = 5\overline{g} - 2D^2 - 1$.

We use the next lemma.

Lemma 5. *If $\sigma \geq 13$ then $5\overline{g} - 2D^2 - 1 \geq 0$; hence $2\omega \geq g$.*

Proof. Since (S, D) is minimal, by Hartshorne's lemma, we have either (1) $|2D + \sigma K_S| \neq \emptyset$ or (2) $B = 1$ and $|3D + eK_S| \neq \emptyset$.

In case (1), we have $(2D + \sigma K_S) \cdot D = 2\sigma\overline{g} - (\sigma - 2)D^2 \geq 0$ and then

$$5\overline{g} - 2D^2 \geq \frac{\sigma - 10}{2\sigma}D^2.$$

We distinguish the various cases according to the signature of D^2.

(i) If $D^2 < 0$ then $5\overline{g} - 2D^2 \geq 5\overline{g} \geq 5$. Hence,

$$2\omega - \overline{g} = 5\overline{g} - 2D^2 \geq 1.$$

(ii) If $D^2 = 0$ then $5\overline{g} - 2D^2 = 5\overline{g} \geq 5$. Hence,

$$2\omega - \overline{g} = 2\omega - \overline{g} = 5\overline{g} - 2D^2 = 5\overline{g} > 0.$$

(iii) If $D^2 > 0$ then

$$5\overline{g} - 2D^2 \geq \frac{\sigma - 10}{2\sigma} D^2 > 0.$$

In case (2), we have $(3D + eK_S) \cdot D = 2e\overline{g} - (e-3)D^2 \geq u + \nu_1 > 0$ and then

$$5\overline{g} - 2D^2 > \frac{e - 15}{2e} D^2.$$

By $e - \sigma = \nu_1 + u \geq \nu_1$, we get $e \geq \sigma + 2 \geq 16$.
Hence, we are done. Q.E.D.

12.1. **final case.**
We shall show that when $\sigma \leq 12$ and $g > 0$,

$$\sigma \leq \omega^2 + \omega + 2.$$

Actually, if $\omega \geq 3$, then $\omega^2 + \omega + 2 \geq 14$.
However,if $\omega = 2$ then $\omega^2 + \omega + 2 = 8$ and by the list of types with $\omega \leq 2$ in the appendix, we obtain $\sigma = 8$ if $g > 1$.
Last, if $\omega = 1$ then the type turns out to be $[7 * 9, 1; 1]$.
Note that if $\omega^2 + 3\omega + 2 = \sigma$ then $2g\omega_1 + \overline{g}^2 + \overline{g} = 0$; hence either 1) $g = 0$ or 2) $g = 1$ and $\omega_1 = 0$. In the last case, $\omega_1^2 + \omega_1 + 2 + 2\overline{g} = 2 \geq \sigma$, which contradicts the hypothesis saying $\sigma \geq 7$. Hence, the proof in the case when $g = 0$ is complete.

13. Matsuda's inequality

Replacing ω_1 by $\alpha - 2\overline{g}$, from $\sigma \leq \omega_1^2 + \omega_1 + 2 + 2\overline{g}$, we obtain

$$\sigma \leq \alpha^2 + (1 - 4\overline{g})\alpha + 4\overline{g}^2 + 2. \tag{25}$$

Then

$$\begin{aligned} &\alpha^2 + 5\alpha + 6 - (\alpha^2 + (1 - 4\overline{g})\alpha + 4\overline{g}^2 + 2) \\ &= 4g(\alpha + 1 - \overline{g}) \\ &= 4g(\omega + 1) \\ &\geq 8g, \end{aligned}$$

since $\alpha - \overline{g} = \omega \geq 1$.
Therefore, we get

$$\alpha^2 + 5\alpha + 6 \geq \sigma + 8g \geq \sigma,$$

FIGURE 2

provided that $\sigma \geq 7$.

13.1. **case in which** $g > 0$**.** We shall show that if $g > 0$ then $\sigma \leq \alpha^2+\alpha+2$. This was first proved by Matsuda ([9]).

As a matter of fact, whenever $\overline{g} = g - 1 \geq 0$, we get

$$\alpha^2 + \alpha + 2 - (\alpha^2 + (1 - 4\overline{g})\alpha + 4\overline{g}^2 + 2) = 4\overline{g}(\alpha - \overline{g}) = 4\overline{g}\omega \geq 0.$$

Hence, by Theorem 4,

$$\alpha^2 + \alpha + 2 \geq \alpha^2 + (1 - 4\overline{g})\alpha + 4\overline{g}^2 + 2 \geq \sigma.$$

Thus we obtain

Theorem 7. *Assuming that* $\sigma \geq 7$*, we obtain*

$$\alpha^2 + 5\alpha + 6 \geq \sigma.$$

If the equality holds, then $g = 0$*. Moreover, if* $g > 0$ *then*

$$\alpha^2 + \alpha + 2 \geq \sigma.$$

TABLE 24

ξ	$\widetilde{Z}$						
2	(1^2)	(2)					
	$2\nu_1-2$	$2\nu_1-4$					
3	(1^3)	$(1,2)$	(3)				
	$3\nu_1-3$	$3\nu_1-5$	$3\nu_1-9$				
4	(1^4)	$(1^2,2)$	(2^2)	$(1,3)$	(4)		
	$4\nu_1-4$	$4\nu_1-6$	$4\nu_1-8$	$4\nu_1-10$	$4\nu_1-16$		
5	(1^5)	$(1^3,2)$	$(1,2^2)$	$(1^2,3)$	$(2,3)$	$(1,4)$	(5)
	$5\nu_1-5$	$5\nu_1-7$	$5\nu_1-9$	$5\nu_1-11$	$5\nu_1-13$	$5\nu_1-17$	$5\nu_1-25$
6	(1^6)	$(1^4,2)$	$(1^2,2^2)$	(2^3)	$(1^3,3)$	$(1,2,3)$	$(1^2,4)$
	$6\nu_1-6$	$6\nu_1-8$	$6\nu_1-10$	$6\nu_1-12$	$6\nu_1-12$	$6\nu_1-14$	$6\nu_1-18$
6	(3^2)	$(2,4)$	$(1,5)$	(6)			
	$6\nu_1-18$	$6\nu_1-20$	$6\nu_1-26$	$6\nu_1-36$			

14. PAIRS WITH $\omega \le 4$

14.1. **case in which $\nu_1 \le 3$.** As before $\sigma \ge 7$ is assumed.
If $\nu_1 \le 3$, then $\omega = \frac{\tau_3}{2} - 9 + t_2$.

Moreover, if $\sigma \ge 8$ and $\nu_1 \le 3$, then $\omega = \frac{\tau_3}{2} - 9 + t_2 \ge \frac{(\sigma-3)(\widetilde{B}-6)}{2} - 9 \ge 6$.
If $\sigma = 8$ and $\omega = 6$, then the type is $[8*10, 1:1]$.
If $\sigma = 7$, then $\nu_1 \le 2\sigma/2$; hence $\nu_1 \le 3$ and $\omega \ge 1$.
Furthermore,

- if $\omega = 1$ then the type is $[7*9, 1; 1]$;
- if $\omega = 2$ then the type is $[7*9, 1; 2]$;
- if $\omega = 3$ then the type is $[7*9, 1; 2^2]$;
- if $\omega = 4$ then the type is $[7*9, 1; 2^3]$;
- if $\omega = 5$ then the type is either $[7*9, 1; 2^4]$ or $[7*10, 1; 1]$.

In that follows we assume that $\nu_1 \ge 4$.

Here, assuming $\omega = 2, 3, 4$, we shall determine the types of pairs (S, D). First,we note that $B \le 2$ by Proposition 5 saying that $\omega \ge 12$ if $B \ge 3$.

14.2. **case in which $\lambda \ge 1$.**
First, we suppose that $\lambda = k - \omega_1 \ge 1$.

14.3. **case in which $\lambda \ge 1$ and $p \ge 1$.**
Assume that $\lambda \ge 1$. Thus

$$\nu_1 \le \frac{-\tilde{k} - \omega + 3\overline{g}}{\lambda}.$$

By $\nu_1 \ge 4$, we get

$$4\lambda = 4k - 4\omega + 4\overline{g} \le -\tilde{k} - \omega + 3\overline{g};$$

hence,

$$\overline{g} \leq 3\omega - (4k + \tilde{k}). \tag{26}$$

If $p \geq 1$, then by the formula (28), $\overline{g} \leq 3\omega - 13$. So, from $\overline{g} \geq -1$, it follows that $\omega \geq 4$. Furthermore, if $\omega = 4$, then $\overline{g} = -1$. However, by (4),

$$-\omega_1 + 2\overline{g} \geq \lambda + p(k - 2p) \geq 0.$$

Hence,if $\omega = 4$ then $2\overline{g} \geq \omega_1 = 4 - \overline{g}$. Thus, $3\overline{g} \geq 4$, which contradicts $\overline{g} = -1$.

Therefore, if $p \geq 1$ and $\lambda \geq 1$, then $\omega \geq 5$.

14.4. **case in which $\lambda \geq 1$ and $p = 0, u \geq 1$.** Assume that $p = 0$. Then $k = 2u, \lambda = 2u - \omega + \overline{g}$ and so $\overline{g} = \lambda - 2u + \omega$. Hence,

$$\begin{aligned}\nu_1 &\leq \frac{3\overline{g} - \omega}{\lambda} \\ &= \frac{2\omega - 6u + 3\lambda}{\lambda} \\ &= \frac{2\omega - 6u}{\lambda} + 3.\end{aligned}$$

By $\nu_1 \geq 4$, we get

$$4 \leq \nu_1 \leq \frac{2\omega - 6u}{\lambda} + 3.$$

Hence,

$$\omega - 3u > 0. \tag{27}$$

Therefore,

$$\nu_1 \leq 2\omega - 6u + 3.$$

Accordingly,

$$\sigma \leq 4\omega - 12u + 6. \tag{28}$$

Then by the formula 28,

$$8 \leq \sigma \leq 4\omega - 12u + 6.$$

If $\omega = 2$ or 3 or 4 then $\omega = 4$ and $u = 1$.

Thus, $Y = 8\nu_1 + 2u + \omega_1 = 8\nu_1 + 2 + 4 - \overline{g}$. Moreover, $\sigma = 8$ or 10. Hence, $\nu_1 = 4$ or 5.

By $\lambda = 2u - 4 + \overline{g} = \overline{g} - 2 \geq 1$, we have $\overline{g} \geq 3$.

By

$$\widetilde{\mathcal{Z}} = -2\nu_1 + (\nu_1 - 1)(4 - \overline{g}) + 2\overline{g},$$

if $\nu_1 = 4$ then $\widetilde{\mathcal{Z}} = 4 - \overline{g} \geq 0$. Hence, $\overline{g} = 4$ or 3.

Moreover, $\overline{g} = 4$ implies that $\widetilde{\mathcal{Z}} = 0$ and so $Y = 8\nu_1 + 2 = r\nu_1 = 4r$, a contradiction.

$\overline{g} = 3$ implies that $\widetilde{\mathcal{Z}} = 1$. But $\widetilde{\mathcal{Z}} = 3t_3 + 4t_2 = 1$, which is absurd.

If $\nu_1 = 5$ then $\widetilde{\mathcal{Z}} = 6 - 2\overline{g} \geq 0$. Hence, $\overline{g} = 3$, $\widetilde{\mathcal{Z}} = 0$, which induces that $Y = 8\nu_1 + 3 = r\nu_1$, that is absurd.

14.5. **case in which $\lambda \geq 1$ and $k = 0$.**
Then $\lambda = k - \omega_1 \geq 1$;hence, $\overline{g} \geq 1 + \omega$.
Recall the formula

$$\widetilde{\mathcal{Z}} = 2\overline{g} + (\nu_1 - 1)\omega_1.$$

By $\nu_1 \geq 4$ and $\omega_1 = \omega - \overline{g} \leq -1$, we obtain

$$0 \leq \widetilde{\mathcal{Z}} \leq 2\overline{g} + 3\omega_1 = 3\omega - \overline{g}.$$

Therefore,

$$3\omega \geq \overline{g}. \tag{29}$$

14.5.1. *case in which $\omega = 2$.*
Suppose that $\omega = 2$. Then

$$6 = 3\omega \geq \overline{g} \geq \omega + 1 = 3.$$

We shall distinguish the various cases according to the value of $\overline{g}$.

(1) If $\overline{g} = 3$, then $\omega_1 = -1$ and so $\sigma \leq \omega_1^2 + \omega_1 + 2g = 8$. But $\nu_1 \geq 4$ was assumed and so we get $\nu_1 = 4$ and $\sigma = 8$.
Since $\widetilde{\mathcal{Z}} = 7 - \nu_1 \geq 1$, it follows that $\widetilde{\mathcal{Z}} = 7 - \nu_1 \geq \nu_1 - 1$; thus $8 \geq 2\nu_1$ and so $\nu_1 = 4$. By $\widetilde{\mathcal{Z}} = 7 - \nu_1 = 3 = 3t_3 + 4t_2$, we get $t_3 = 1, t_2 = 0$.
Since $Y' = s\nu_1 - 1 = 3$, it follows that $s = 1$ and so the type is $[8*8; 4^7, 3]$.

(2) If $\overline{g} = 4$, then $\omega_1 = -2$ and so $\widetilde{\mathcal{Z}} = 10 - 2\nu_1 \geq 0$; thus $\nu_1 = 4$ or 5.
If $\nu_1 = 5$ then $\widetilde{\mathcal{Z}} = Y' = 0$ and so by $Y' = s\nu_1 - 2 = 5s - 2$, we arrive at a contradiction.
If $\nu_1 = 4$ then $\widetilde{\mathcal{Z}} = 2 = 3t_3 + 4t_2 = 0$, which has no solution.

(3) If $\overline{g} = 5$, then $\omega_1 = -3$ and so $\widetilde{\mathcal{Z}} = 13 - 3\nu_1 \geq \nu_1 - 1$; thus $\nu_1 < 4$.

(4) If $\overline{g} = 6$, then $\omega_1 = -4$ and so $\widetilde{\mathcal{Z}} = 16 - 4\nu_1 \geq 0$; thus $\nu_1 = 4$ and $\widetilde{\mathcal{Z}} = 0$. Hence, s=1 and the type becomes $[8 * 8; 4^7]$.

14.5.2. *case in which $\omega = 3$.*
Suppose that $\omega = 3$. Then

$$9 = 3\omega \geq \overline{g} \geq \omega + 1 = 4.$$

Furthermore,

$$\widetilde{\mathcal{Z}} = 2\overline{g} + (\nu_1 - 1)(3 - \overline{g}) = 3\overline{g} - 3 + (3 - \overline{g})\nu_1.$$

We shall distinguish the various cases according to the value of $\overline{g}$.

(1) If $\overline{g} = 4$ then $\omega_1 = -1$ and

$$\widetilde{\mathcal{Z}} = 9 - \nu_1.$$

If $9 = \nu_1$ then $\widetilde{\mathcal{Z}} = 0$; hence, $Y' = 0$. But $Y = 8\nu_1 - 1 > 0$, a contradiction. Thus, $9 - \nu_1 \geq \nu_1 - 1$. Hence, $\nu_1 = 4\text{or}5$.

If $\nu_1 = 5$ then $Y' = s\nu_1 - 1 = 4$ and so $s = 1$. The type is $[10 * 10; 5^7, 4]$.

If $\nu_1 = 4$ then $\widetilde{\mathcal{Z}} = 12 - 8 = 4 = 3t_3 + 4t_2$. Thus $t_3 = 0, t_2 = 1$. $Y' = 4s - 1 = 2$, a contradiction.

(2) If $\overline{g} = 5$ then $\omega_1 = -2$ and

$$\widetilde{\mathcal{Z}} = 12 - 2\nu_1.$$

If $\widetilde{\mathcal{Z}} = 0$ then $\nu_1 = 6, Y' = 0$. Hence, $g = 11^2 - 15r = 6$, which is impossible.

Otherwise, $\widetilde{\mathcal{Z}} = 12 - 2\nu_1 \geq \nu_1 - 1$ then $\nu_1 = 4$

By $\widetilde{\mathcal{Z}} = 12 - 2\nu_1 = 4 = 3t_3 + 4t_2$, we get $t_2 = 1$ and so $Y' = s\nu_1 - 2 = 2$ and so $s = 1$. The type becomes $[8 * 8; 4^7, 2]$.

(3) If $\overline{g} = 6$ then $\omega_1 = -3$ and

$$\widetilde{\mathcal{Z}} = 15 - 3\nu_1 \geq 0,$$

which implies $\nu_1 = 4$ or 5.

If $\nu_1 = 4$ then $\widetilde{\mathcal{Z}} = 15 - 3\nu_1 = 3$. Hence, $t_3 = 1$ and so $Y' = 4s - 3 = s\nu_1 - 3 = 3$ and so $4s = 6$, a contradiction.

If $\nu_1 = 5$ then $\widetilde{\mathcal{Z}} = 0$; thus $Y' = 0$ and $Y' = s\nu_1 - 3 = 0$, contradiction.

(4) If $\overline{g} = 7$ then $\omega_1 = -4$ and

$$\widetilde{\mathcal{Z}} = 18 - 4\nu_1 \geq \nu_1 - 1,$$

which implies $\nu_1 < 4$, a contradiction.

(5) If $\overline{g} = 8$ then $\omega_1 = -5$ and

$$\widetilde{\mathcal{Z}} = 21 - 5\nu_1 \geq \nu_1 - 1,$$

which implies $\nu_1 < 4$, a contradiction.

(6) If $\overline{g} = 9$ then $\omega_1 = -6$ and

$$\widetilde{\mathcal{Z}} = 24 - 6\nu_1 \geq \nu_1 - 1,$$

which implies $\nu_1 < 4$, a contradiction.

14.5.3. *case in which* $\omega = 4$.

Suppose that $\omega = 4$. Then

$$12 = 3\omega \geq \overline{g} \geq \omega + 1 = 5.$$

Furthermore,

$$\widetilde{\mathcal{Z}} = 2\overline{g} + (\nu_1 - 1)(4 - \overline{g}) = 3\overline{g} - 4 + (4 - \overline{g})\nu_1.$$

We shall distinguish the various cases according to the value of $\overline{g}$.

(1) If $\overline{g} = 5$ then $\omega_1 = -1$ and

$$\widetilde{\mathcal{Z}} = 11 - \nu_1.$$

But by $Y' = 11s - 1 > 0$, we have $\widetilde{\mathcal{Z}} > 0$; thus $\widetilde{\mathcal{Z}} \geq \nu_1 - 1$. Hence, $\nu_1 \leq 6$.

If $\nu_1 = 6$ then $\widetilde{\mathcal{Z}} = 11 - \nu_1 = 5 = 5t_5 + 8t_4 + \cdots$. Hence, $t_5 = 1$ and $Y' = 6s - 1 = 5$. Hence, $s = 1$ and the type becomes $[12 * 12; 6^7, 5]$.

If $\nu_1 = 5$ then $\widetilde{\mathcal{Z}} = 11 - \nu_1 = 6 = 4t_4 + 6t_3 + \cdots$. Hence, $t_3 + t_2 = 1$, which implies that $Y' = 2$ or 3. But $Y' = 5s - 1 \geq 4$, a contradiction.

If $\nu_1 = 4$ then $\widetilde{\mathcal{Z}} = 11 - \nu_1 = 7 = 3t_3 + 4t_2$; hence, $t_3 = t_2 = 1$. Therefore, $Y' = 5$. However, $Y' = 4s - 1$; a contradiction.

(2) If $\overline{g} = 6$ then $\omega_1 = -2$ and

$$\widetilde{\mathcal{Z}} = 14 - 2\nu_1 \geq 0.$$

Hence, $\nu_1 \leq 7$ and if $\nu_1 = 7$ then $Y' = 0$. But $Y' = s\nu_1 - 2 = 0$, a contradiction.

But $\widetilde{\mathcal{Z}} = 14 - 2\nu_1 \geq \nu_1 - 1$, which implies that $\nu_1 \leq 5$.

If $\nu_1 = 5$ then $\widetilde{\mathcal{Z}} = 14 - 2\nu_1 = 4 = 4t_4 + 6t_3 + 6t_2$. Thus $t_4 = 1$ and $Y' = s\nu_1 - 2 = 5s - 2 = 4$, a contradiction.

If $\nu_1 = 4$ then $\widetilde{\mathcal{Z}} = 14 - 2\nu_1 = 6 = 3t_3 + 4t_2$. Thus $t_3 = 2$ and $Y' = s\nu_1 - 2 = 6$; we get $s\nu_1 = 4s = 8$. Hence, $s = 2$ and the type becomes $[8 * 8; 4^6, 3^2]$.

(3) If $\overline{g} = 7$ then $\omega_1 = -3$ and

$$\widetilde{\mathcal{Z}} = 17 - 3\nu_1 \geq \nu_1 - 1.$$

Hence, $\nu_1 \leq 4$ and $\nu_1 = 4$.

But $\widetilde{\mathcal{Z}} = 5 = 3t_3 + 4t_2$,which has no solution.

(4) If $\overline{g} = 8$ then $\omega_1 = -4$ and

$$\widetilde{\mathcal{Z}} = 20 - 4\nu_1 \geq 0.$$

Hence, $\nu_1 \leq 5$ and if $\nu_1 = 5$ then $Y' = 5s - 4 = 0$, a contradiction.

If $\nu_1 = 4$ then

$$\widetilde{\mathcal{Z}} = 20 - 4\nu_1 = 4 = 3t_3 + 4t_2.$$

Hence, $t_2 = 1$ and then $Y' = 4s - 4 = 2$,a contradiction.

(5) If $\overline{g} = 9$ then $\omega_1 = -5$ and

$$\widetilde{\mathcal{Z}} = 23 - 5\nu_1 \geq 0.$$

Hence, $\nu_1 = 4$.

$$\widetilde{\mathcal{Z}} = 23 - 5\nu_1 = 3 = 3t_3 + 4t_2.$$

Hence, $t_3 = 1$ and then $Y' = 4s - 5 = 3$. Thus $s = 2$ is derived. The type becomes $[8 * 8; 4^6, 3]$.

(6) If $\overline{g} = 10$ then $\omega_1 = -6$. and then

$$\widetilde{\mathcal{Z}} = 26 - 6\nu_1 \geq 0.$$

Hence, $\nu_1 = 4$.

$$\widetilde{\mathcal{Z}} = 26 - 6\nu_1 = 2 = 3t_3 + 4t_2.$$

This has no solution.

(7) If $\overline{g} = 11$ then $\omega_1 = -7$ and

$$\widetilde{\mathcal{Z}} = 29 - 7\nu_1 \geq 0.$$

Hence, $\nu_1 = 4$.

$$\widetilde{\mathcal{Z}} = 1 = 3t_3 + 4t_2.$$

This has no solution.

(8) If $\overline{g} = 12$ then $\omega_1 = -8$ and

$$\widetilde{\mathcal{Z}} = 32 - 8\nu_1 \geq 0.$$

Hence, $\nu_1 = 4$.

$$\widetilde{\mathcal{Z}} = 0.$$

Hence, $Y' = s\nu_1 - 8 = 4s - 8 = (r - t)4$.
Therefore,$r = t + s - 2 = 8 - 2 = 6$. The type becomes $[8 * 8; 4^6]$.

14.6. **case in which $\lambda \leq 0$ and $p \geq 1$.**
Given $\omega_1 \geq k \geq wp \geq 3$ and $p \geq 1$, one has $\omega \geq 3 + \overline{g}$.

14.6.1. *case in which $\omega = 2$.*
Then $g = u = 0$ and so

- $X = 8\nu_1^2 + 6\nu_1 + 1 + \omega_1 - 2\overline{g} = 8\nu_1^2 + 6\nu_1 + 6$,
- $Y = 8\nu_1 + 1 + \omega_1 = 8\nu_1 + 6$.

Thus, $\widetilde{\mathcal{Z}} = -6 < 0$, a contradiction.

14.6.2. *case in which $\omega = 3$.*
Then $g = 0, 1$.
If $g = 1$ then $k = 3, u = 0$. Hence,

- $X = 8\nu_1^2 + 6\nu_1 + 1 + \omega_1 - 2\overline{g} = 8\nu_1^2 + 6\nu_1 + 4$,
- $Y = 8\nu_1 + 1 + \omega_1 = 8\nu_1 + 6$.

Thus, $\widetilde{\mathcal{Z}} = -4 < 0$, a contradiction.

If $g = 0$ then $k = w, u = 0$ and so

- $X = 8\nu_1^2 + 2w\nu_1 + w + 4$,
- $Y = 8\nu_1 + w + 4$.

Thus, $\widetilde{\mathcal{Z}} = (4-w)\nu_1 - w - 4 \geq 0$. Hence, $w = 3$ and $\widetilde{\mathcal{Z}} = \nu_1 - 7$.

If $\widetilde{\mathcal{Z}} = 0$ then $\nu_1 = 7$ and $Y = 8\nu_1 + 7 = r\nu_1$. Hence, $r = 9$ and the type turns out to be $[15 * 22, 1; 7^9]$.

Otherwise, $\widetilde{\mathcal{Z}} = \nu_1 - 7 \geq \nu_1 - 1$, a contradiction.

14.6.3. *case in which* $\omega = 4$.

Then $g = 0, 1, 2$. We distinguish the following cases according to g.

(1) If $g = 0$ then $\omega_1 = 5$. Hence,

- $X = 8\nu_1^2 + 2w\nu_1 + w - 2 + 7$,
- $Y = 8\nu_1 + w + 5$.

Thus, $\widetilde{\mathcal{Z}} = (5-w)\nu_1 - 5 - w$.

If $w = 3$ then $\widetilde{\mathcal{Z}} = 2\nu_1 - 8 \geq 0$.

Suppose that $\widetilde{\mathcal{Z}} = 0$, i.e. $\nu_1 = 4$. Then $Y = 8\nu_1 + w + 5 = 8\nu_1 + 8 = r\nu_1$. From $8 = (r-8)\nu_1$, it follows that $r = 10, \nu_1 = 4$. Hence, the type becomes $[9 * 13, 1; 4^{10}]$.

Otherwise, $\widetilde{\mathcal{Z}} = 2\nu_1 - 8 \geq \nu_1 - 1$. Then $\nu_1 < 4$, a contradiction.

If $w = 4$ then $\widetilde{\mathcal{Z}} = \nu_1 - 9$.

Suppose that $\widetilde{\mathcal{Z}} = 0$, i.e. $\nu_1 = 9$. Then $Y = 8\nu_1 + 9 = r\nu_1$. From $9 = (r-8)\nu_1$, it follows that $r = 9, \nu_1 = 9$. Hence, the type becomes $[19 * 19; 9^9]$.

Otherwise, $\widetilde{\mathcal{Z}} = \nu_1 - 9 \geq \nu_1 - 1$, a contradiction.

(2) If $g = 1$ then $\omega_1 = 4$. Hence, $k = w \leq \omega_1 = 4$.

- $X = 8\nu_1^2 + 2w\nu_1 + w - 2 + 4$,
- $Y = 8\nu_1 + w + 4$.

Thus, $\widetilde{\mathcal{Z}} = (4-w)\nu_1 - 2 - w \geq 0$. Then $w = 3$ and $\widetilde{\mathcal{Z}} = \nu_1 - 5$. Hence, $\nu_1 = 5$ and $Y = 8\nu_1 + 7 = r\nu_1$, a contradiction.

(3) If $g = 2$ then $\omega_1 = 3$. Hence, $k = w \leq \omega_1 = 3$. Hence $w = 3$ and

- $X = 8\nu_1^2 + 6\nu_1 + 2$,
- $Y = 8\nu_1 + 6$.

Thus, $\widetilde{\mathcal{Z}} = -2 \geq 0$, a contradiction.

14.7. **case in which $\lambda \leq 0$ and $p = 0, u \geq 1$.**

Given $\omega_1 \geq k = 2u \geq 2$, one has $\omega \geq 2u + \overline{g}$. Moreover,

- $X = 8\nu_1^2 + 4u\nu_1 + \omega_1 - 2\overline{g}$,
- $Y = 8\nu_1 + 2u + \omega_1$.

Thus, $\widetilde{\mathcal{Z}} = -2u\nu_1 + (\nu_1 - 1)\omega_1 + 2\overline{g}$.

14.7.1. *case in which* $\omega = 2$.

If $\omega = 2$ then by $\omega = 2 \geq 2u + \overline{g}$, we get $u = 1$ and $\overline{g} = -1, 0$. Hence, $\widetilde{\mathcal{Z}} = -2\nu_1 + (\nu_1 - 1)(2 - \overline{g}) + 2\overline{g}$.

If $\overline{g}=-1$ then $\widetilde{\mathcal{Z}}=\nu_1-5$. In this case, $\nu_1=5$ and thus $Y=8\nu_1+2+3=r\nu_1$. Hence, $r=9$ and the type becomes $[10*11;5^9]$.

If $\overline{g}=0$ then $\widetilde{\mathcal{Z}}=-2$, a contradiction.

14.7.2. *case in which* $\omega=3$.

If $\omega=3$ then $u=1$ and $\overline{g}=-1,0,1$.

We distinguish the various cases according to g.

- If $\overline{g}=-1$ then $\omega=4$ and $\widetilde{\mathcal{Z}}=2\nu_1-6\geq 2$, which is impossible.
- If $\overline{g}=0$ then $\omega=3$ and $\widetilde{\mathcal{Z}}=\nu_1-3\geq 1$, which is impossible.
- If $\overline{g}=1$ then $\omega=2$ and $\widetilde{\mathcal{Z}}=0$. Since $Y=8\nu_1+2+2=r\nu_1$, it follows that $4=(r-8)\nu_1$. Hence, $r=9$ and $\nu_1=4$ and the type becomes $[8*9;5^9]$.

14.7.3. *case in which* $\omega=4$.

If $\omega=4$ then $\omega=4\geq 2u+\overline{g}$.

Furthermore, if $u=2$ then $\overline{g}=-1,0$. If $u=1$ then $\overline{g}=-1,0,1,2$.

We distinguish the various cases according to u and g.

(1) If $u=2$ and $\overline{g}=-1$, then

$$\widetilde{\mathcal{Z}}=(5-2u)\nu_1-7=\nu_1-7.$$

If $\nu_1=7$ then $\widetilde{\mathcal{Z}}=0$ and $Y=8\nu_1+9=r\nu_1$, a contradiction. Otherwise, $\widetilde{\mathcal{Z}}=\nu_1-7\geq\nu_1-1$, a contradiction.

(2) If $u=2$ and $\overline{g}=0$, then $\widetilde{\mathcal{Z}}=-4$, a contradiction.

(3) If $u=1$ and $\overline{g}=-1$, then $\omega_1=5$ and

$$\widetilde{\mathcal{Z}}=3\nu_1-7.$$

If $t_{\nu_1-1}=1$ then $\widetilde{\mathcal{Z}}-(\nu_1-1)=2\nu_1-6$. In this case, the equation $2\nu_1-6=2(\nu_1-2)t_{\nu_1-2}+\cdots$ has no solution. If $t_{\nu_1-1}=0$ then $\widetilde{\mathcal{Z}}=(2\nu_1-4)t_{\nu_1-2}+\cdots$.

If there exist at least two positive t_{ν_1-j}, then there exists an integer j such that $\widetilde{\mathcal{Z}}\geq 2j(\nu_1-j)$ where $\nu_1-j\geq 2$.

Then

$$j+1+\frac{j-4}{2j-3}\geq\nu_1\geq j+2.$$

Hence $-1\geq j$, a contradiction.

However, from $3\nu_1-7=j(\nu_1-j)$, it follows that

$$j+3+\frac{2}{j-3}=\nu_1.$$

Hence, $j=5$ or 4. In both cases, $\nu_1=9$. But,

$$Y=8\nu_1+2+5=t\nu_1+\nu_1-j.$$

Then $7+j=(t-7)\nu_1=9(t-7)$. Recalling that $7+j=12$ or 11, we arrive at a contradiction.

(4) If $u = 1$ and $\overline{g} = 0$, then $\omega_1 = 4$ and

$$\widetilde{\mathcal{Z}} = 2\nu_1 - 4.$$

Then $t_{\nu_1 - 2} = 1$ and

$$Y = 8\nu_1 + 2 + 4 = t\nu_1 + \nu_1 - 2.$$

Hence, $8 = (t - 7)\nu_1$. Thus we have two cases:

- $\nu_1 = 8, t = 7$, where the type is $[16 * 17; 8^8, 6]$;
- $\nu_1 = 4, t = 9$, where the type is $[8 * 9; 4^9, 2]$.

(5) If $u = 1$ and $\overline{g} = 1$, then $\omega_1 = 3$ and

$$\widetilde{\mathcal{Z}} = \nu_1 - 1.$$

Then $t_{\nu_1 - 1} = 1$ and

$$Y = 8\nu_1 + 2 + 3 = t\nu_1 + \nu_1 - 1.$$

Hence, $6 = (t - 7)\nu_1$. Then $t = 8$ and $\nu_1 = 6$. The type becomes $[12 * 13; 6^8, 5]$.

(6) If $u = 1$ and $\overline{g} = 2$, then $\omega_1 = 2$ and

$$\widetilde{\mathcal{Z}} = 2 \geq \nu_1 - 1.$$

Hence, $\nu_1 < 4$, a contradiction.

14.8. **case in which $\lambda \leq 0$ and $k = 0$.**
In this case, $p = 0, u = 0$ and so $\overline{g} \leq \omega$. We obtain the fundamental equalities:

- $X' = s\nu_1^2 + \omega_1 - 2\overline{g}$,
- $Y' = s\nu_1 + \omega_1$.

Then $\widetilde{\mathcal{Z}} = (\nu_1 - 1)\omega_1 + 2\overline{g}$.
We shall use the following symbol:

- $\varepsilon(t) = \sum_{j=1}^{\nu_1 - 2} t_{\nu_1 - j}$,

14.8.1. *case in which* $\omega = 2$. Then by $\omega_1 = \omega - \overline{g} = 2 - \overline{g} \geq 0$, we see that $\overline{g} = -1, 0, 1, 2$.
We shall distinguish the various cases according to the value of $\overline{g}$.

(1) $\overline{g} = -1$. Then $\omega_1 = 3$ and

$$\widetilde{\mathcal{Z}} = 3\nu_1 - 5.$$

(i) First assume that $t_{\nu_1 - 1} = 1$. If $\varepsilon(t) = 2$, then we find j such that $\widetilde{\mathcal{Z}} - (\nu_1 - 1) = 2\nu_1 - 4 = j(\nu_1 - j)$. Solving this we have $j = 2$ or $j + 2 = \nu_1$. Therefore, $Y' = s\nu_1 + 3 = \nu_1 - 1 + 2$ or $Y' = s\nu_1 + 3 = \nu_1 - 1 + \nu_1 - 2$.
However the former case does not occur. In the last case, $6 = (2 - s)\nu_1$. Thus $s = 1$ and $\nu = 6$. The type becomes $[12 * 12; 6^7, 5, 4]$.

If $\varepsilon(t) \geq 3$, then there exists a number j such that $2\nu_1 - 4 \geq 2j(\nu_1 - j)$, where $\nu_1 - j \geq 2$. Then we get

$$j + 1 - \frac{1}{j-1} \geq \nu_1 \geq j + 2.$$

This is a contradiction.

(ii) Assume that $t_{\nu_1 - 1} = 0$. Then by $Y' = s\nu_1 + \omega_1 > \nu_1$, $\varepsilon(t) \geq 3$ and so there exists j such that $3\nu_1 - 5 \geq 2j(\nu_1 - j)$, where $\nu_1 - j \geq 2$. Then

$$j + 1 + \frac{j-2}{2j-3} \geq \nu_1 \geq j + 2.$$

We have $\frac{j-2}{2j-3} \geq 1$, which induces $1 \geq j$, a contradiction.

(iii) Assume that $t_{\nu_1 - 1} = 2$.

Then

$$\widetilde{\mathcal{Z}} - 2(\nu_1 - 1) = \nu_1 - 3 \geq 2(\nu_1 - 2).$$

Thus, $2 > \nu_1$, a contradiction.

It is easy to derive a contradiction from $t_{\nu_1 - 1} > 2$.

(2) $\overline{g} = 0$. Then $\omega_1 = 2$ and

$$\widetilde{\mathcal{Z}} = 2\nu_1 - 2 = (\nu_1 - 1)t_{\nu_1 - 1} + 2(\nu_1 - 2)(t_{\nu_1 - 2} + t_2) + \cdots .$$

Thus $t_{\nu_1 - 1} = 2$ and $Y' = 2\nu_1 - 2 = s\nu_1 + 2$. From this it follows that $4 = (2 - s)\nu_1$. Hence, $s = 1$ and $\nu_1 = 4$. The type becomes $[8 * 8; 4^7, 3^2]$.

(3) $\overline{g} = 1$. Then $\omega_1 = 1$ and by the inequality (2), we get

$$\sigma \leq \omega_1^2 + \omega_1 + 2 + 2\overline{g} = 2 + 2 + 2.$$

This contradicts the hypothesis that $\sigma \geq 7$.

(4) $\overline{g} = 2$. Then $\omega_1 = 0$ and by the inequality (2), we get

$$\sigma \leq \omega_1^2 + \omega_1 + 2 + 2\overline{g} = 2 + 4.$$

This contradicts the hypothesis that $\sigma \geq 7$.

14.8.2. *case in which* $\omega = 3$. Then by $\omega - \overline{g} = 3 - \overline{g} \geq 0$, we see that $\overline{g} = -1, 0, 1, 2$.

We distinguish the various cases according to the value of $\overline{g}$.

(1) $\overline{g} = -1$. Then $\omega_1 = 4$ and

$$\widetilde{\mathcal{Z}} = 4\nu_1 - 6.$$

(i) First assume that $t_{\nu_1 - 1} = 1$.

If $\varepsilon(t) = 2$, then we can find j such that $\widetilde{\mathcal{Z}} - (\nu_1 - 1) = 3\nu_1 - 5 = j(\nu_1 - j)$. Thus

$$j + 3 + \frac{4}{j-3} = \nu_1.$$

From this we obtain the next table:

TABLE 25

$j-3$	j	$j+3$	ν_1	ν_1-j	ν_1-1	Y'
4	7	10	11	4	10	14 or 17
2	5	8	10	5	9	14
1	4	7	11	7	10	14 or 17

Recalling that $Y' = s\nu_1 + 4$, we obtain $\nu_1 = 9, s = 1, Y' = 14$. Thus the type becomes $[20 * 20; 10^7, 9, 5]$.

If $\varepsilon(t) \geq 3$, then we find j such that $\widetilde{\mathcal{Z}} - (\nu_1 - 1) = 3\nu_1 - 5 \geq 2j(\nu_1 - j)$. Thus $2j^2 - 5 \geq (2j-3)\nu_1$ and so

$$j + 1 - \frac{2}{2j-3} \geq \nu_1 \geq j + 2.$$

This is impossible.

(ii) Assume that $t_{\nu_1 - 1} = 2$.

If $\varepsilon(t) = 2$, then we find $j > 1$ such that

$$\widetilde{\mathcal{Z}} - 2(\nu_1 - 1) = 2\nu_1 - 4 = j(\nu_1 - j).$$

Thus

$$j^2 - 4 = (j-2)\nu_1.$$

If $j > 2$ then $\nu_1 + j + 2$.

If $j = 2$ then $\widetilde{\mathcal{Z}} - 2(\nu_1 - 1) = 2\nu_1 - 4t_{\nu_1 - 2}$. Hence, $t_{\nu_1-2} = 1$ and $Y' = 2(\nu_1 - 1) + \nu_1 - 2 = 3\nu_1 - 4$; thus $Y' = s\nu_1 + 4 = 3\nu_1 - 4$. Hence,

$$8 = (3 - s)\nu_1.$$

We have two cases:

- $s = 1, \nu_1 = 4$. The type becomes $[8 * 8; 4^7, 3^2, 2]$.
- $s = 2, \nu_1 = 8$. The type becomes $[16 * 16; 8^6, 7^2, 6]$.

Moreover, if $j + 2 = \nu_1$ then $\widetilde{\mathcal{Z}} - 2(\nu_1 - 1) = (2\nu_1 - 4)t_2$. Hence, $t_2 = 1$ and $Y' = 2(\nu_1 - 1) + 2 = 2\nu_1$; thus $Y' = s\nu_1 + 4 = 2\nu_1$. Hence, $= (2 - s)\nu_1$. Therefore, $s = 1, \nu_1 = 4$. The type becomes $[8 * 8; 4^7, 3^2, 2]$.

If $\varepsilon(t) > 2$, then we find $j > 1$ such that

$$\widetilde{\mathcal{Z}} - 2(\nu_1 - 1) = 2\nu_1 - 4 \geq 2j(\nu_1 - j).$$

From this we can derive a contradiction.

(iii) Assume that $t_{\nu_1 - 1} = 3$. Then $\widetilde{\mathcal{Z}} - 3(\nu_1 - 1) = \nu_1 - 3$, contradiction.

(iv) Assume that $t_{\nu_1 - 1} = 4$. Then $\widetilde{\mathcal{Z}} - 4(\nu_1 - 1) = -2$, contradiction.

(v) Assume that $t_{\nu_1 - 1} = 0$ and $t_{\nu_1 - 2} = 1$.

If $\varepsilon(t) = 2$, then we find $j > 1$ such that

$$\widetilde{\mathcal{Z}} - 2(\nu_1 - 2) = 2\nu_1 - 2 = j(\nu_1 - j).$$

Thus

$$j+2+\frac{2}{j-2}=\nu_1.$$

From this we obtain the next table:

TABLE 26

$j-2$	j	$j+2$	ν_1	ν_1-j	ν_1-2	Y'
2	4	6	7	3	5	8
1	3	5	7	4	5	9

But $Y'=s\nu_1+4=7s+4$, which is not equal to 8 or 9.
If $\varepsilon(t)>2$, then we find $j>1$ such that

$$\widetilde{\widetilde{Z}}-2(\nu_1-2)=2\nu_1-2\geq 2j(\nu_1-j).$$

From this it is easy to derive a contradiction.

(2) $\overline{g}=0$. Then $\omega_1=3$ and $\widetilde{\widetilde{Z}}=3\nu_1-3$.

(i) First assume that $t_{\nu_1-1}=1$. Then $\widetilde{\widetilde{Z}}-(\nu_1-1)=2(\nu_1-1)$.
If $\varepsilon(t)=2$, then we find $j>1$ such that

$$\widetilde{\widetilde{Z}}-2(\nu_1-2)=2\nu_1-2=j(\nu_1-j).$$

Thus

$$j+2+\frac{2}{j-2}=\nu_1.$$

From this we obtain the next table:

TABLE 27

$j-2$	j	$j+2$	ν_1	ν_1-j	ν_1-1	Y'
2	4	6	7	3	6	9
1	3	5	7	4	6	10

But $Y'=s\nu_1+3=7s+3$, which means $j=3, s=1$.
The type becomes $[14*14;7^7,6,4]$.

(ii) Assume that $t_{\nu_1-1}=2$. Then $\widetilde{\widetilde{Z}}-2(\nu_1-1)=\nu_1-1$. If $\varepsilon(t)=3$, then we can find $j>1$ such that

$$\widetilde{\widetilde{Z}}-2(\nu_1-1)=\nu_1-=j(\nu_1-j).$$

Then $j=1$ or $j=\nu_1-1$, a contradiction.

(iii) Assume that $t_{\nu_1-1}=3$. Then $\widetilde{\widetilde{Z}}-3(\nu_1-1)=0$. Thus $Y'=3(\nu_1-1)$. By the way, from $Y'=s\nu_1+3$, it follows that $s\nu_1+3=3(\nu_1-1)$. Hence, $6=(3-s)\nu_1$. Therefore, $\nu_1=6$ and $s=2$. The type becomes $[12*12;6^6,5^3]$.

(3) $\overline{g} = 1$. Then $\omega_1 = 2$ and $\widetilde{\mathcal{Z}} = 2\nu_1$.

First assume that $t_{\nu_1 - 1} = 1$. Then $\widetilde{\mathcal{Z}} - (\nu_1 - 1) = \nu_1 + 1$.
If $\varepsilon(t) = 2$, then we find $j > 1$ such that

$$\widetilde{\mathcal{Z}} - (\nu_1 - 2) = \nu_1 + 1 = j(\nu_1 - j).$$

Thus

$$j + 1 + \frac{2}{j-1} = \nu_1.$$

From this we obtain the next table:

TABLE 28

$j-1$	j	$j+1$	ν_1	$\nu_1 - j$	$\nu_1 - 1$	Y'
2	3	4	5	2	4	6
1	2	3	5	3	4	7

Thus $\nu_1 = 5$ and $Y' = 5s + 2$, by definition. Then $s = 1$ and the type becomes $[10 * 10; 5^7, 4, 3]$.

(4) $\overline{g} = 2$. Then $\omega_1 = 1$ and $\widetilde{\mathcal{Z}} = \nu_1 + 3$.

First assume that $t_{\nu_1 - 1} = 1$. Then $\widetilde{\mathcal{Z}} - (\nu_1 - 1) = 4$.
If $\varepsilon(t) = 2$, then we find $j > 1$ such that

$$\widetilde{\mathcal{Z}} - (\nu_1 - 1) = 4 = j(\nu_1 - j).$$

Thus $j = 2, \nu_1 = 4$. Hence, $Y' = 3 + 2$. Moreover, $Y' = s\nu_1 + 1 = 4s + 1 = 5$. Hence, $s = 1$. The type becomes $[8 * 8; 4^7, 3, 2]$.

(5) $\overline{g} = 3$. Then $\omega_1 = 0$ and $\widetilde{\mathcal{Z}} = 6$. From $\widetilde{\mathcal{Z}} = 6 = (\nu_1 - 1)t_{\nu_1 - 1} + \cdots$, it follows that $\nu_1 = 4, t_3 = 2$. But $Y' = 4s = s\nu_1 = 3 + 3 = 6$, a contradiction.

14.8.3. *case in which* $\omega = 4$ *and* $g = 0$.
Then by $\omega - \overline{g} = 4 - \overline{g} \geq 0$, we see that $\overline{g} = -1, 0, 1, 2, 3$.
We distinguish the various cases according to the value of $\overline{g}$.

(1) $\overline{g} = -1$. Then $\omega_1 = 5$ and

$$\widetilde{\mathcal{Z}} = 5\nu_1 - 7.$$

(i) First assume that $t_{\nu_1 - 1} = 1$.
If $\varepsilon(t) = 2$, then we find j such that $\widetilde{\mathcal{Z}} - (\nu_1 - 1) = 4\nu_1 - 6 = j(\nu_1 - j)$.
Thus

$$j + 4 + \frac{4}{j-4} = \nu_1.$$

From this we obtain the next table: By $Y' = s\nu_1 + 5$, we conclude that $s = 1$ and $\nu_1 = 15$. Hence, the type becomes $[30 * 30; 15^7, 14, 6]$.

TABLE 29

$j-4$	j	$j+4$	ν_1	ν_1-j	ν_1-1	Y'
10	14	18	19	5	18	23
5	9	13	15	6	14	21
2	6	10	15	9	14	23
1	5	9	19	14	18	32

If $\varepsilon(t) \geq 3$, then we find j such that

$$4\nu_1 - 6 \geq 2j(\nu_1 - j).$$

Hence,

$$j + 2 + \frac{4}{j-1} \geq \nu_1 \geq 2j,$$

and so

$$j + \frac{1}{j-1} \geq j.$$

Thus, $j = 2$.

Therefore, if $\varepsilon(t) = 3$, then we find j such that $\widetilde{\mathcal{Z}} - (\nu_1 - 1) - 2(\nu_1 - 2) = 2\nu_1 - 2 = j(\nu_1 - j)$. Thus

$$j + 2 + \frac{2}{j-2} = \nu_1.$$

From this we obtain the next table:

TABLE 30

$j-2$	j	$j+2$	ν_1	ν_1-j	ν_1-1	ν_1-2	Y'
2	4	6	7	3	6	5	14
1	3	5	7	4	6	5	15

However, $Y' = 7s + 5$, which cannot be 14 or 15. This case does not occur.

Finally, if $\varepsilon(t) \geq 4$, then we find j such that $2\nu_1 - 2 \geq 2j(\nu_1 - j)$. Then $j = 1$ or $j = \nu_1 - 1$, contradiction.

(ii) Assume that $t_{\nu_1 - 1} = 2$.

If $\varepsilon(t) = 3$, then we find j such that $\widetilde{\mathcal{Z}} - 2(\nu_1 - 1) = 3\nu_1 - 5 = j(\nu_1 - j)$. Thus

$$j + 3 + \frac{4}{j-3} = \nu_1.$$

From this we obtain the next table: By $Y' = s\nu_1 + 5$, we conclude that $27 = s\nu_1 + 5, \nu_1 = 11, 2 = 2$. Hence, the type becomes $[22 * 22; 11^6, 10^2, 7]$.

If $\varepsilon(t) \geq 4$, then we find j such that $3\nu_1 - 5 \geq 2j(\nu_1 - j)$. Thus

$$2j^2 - 5 \geq (2j - 3)\nu_1 \geq 2(2j - 3)j.$$

TABLE 31

$j-3$	j	$j+3$	ν_1	$\nu_1 - j$	$\nu_1 - 1$	Y'
4	7	10	11	4	10	24
2	5	8	10	5	9	23
1	4	7	11	7	10	27

Hence,

$$0 \geq 2j^2 - 6j + 5.$$

Then $j < 2$, a contradiction.

(iii) Assume that $t_{\nu_1 - 1} = 3$.

If $\varepsilon(t) = 4$, then we find j such that $\widetilde{\mathcal{Z}} - 3(\nu_1 - 1) = 2\nu_1 - 4 = j(\nu_1 - j)$. Then $j = 2$.

Hence, (a) $t_{\nu_1 - 2} = 1$ or (b) $t_2 = 1$.

In the case (a), we get $Y' = 3(\nu_1 - 1) + \nu_1 - 2$ and $Y' = s\nu_1 + 5$. Hence, $s\nu_1 + 5 = 4\nu_1 - 5$. In other words,

$$10 = (4 - s)\nu_1.$$

Hence, either 1)$\nu_1 = 10, s = 3$, or 2) $\nu_1 = 5, s = 2$.

In case 1), the type becomes $[20 * 20; 10^5, 9^3, 8]$.

Moreover, in case 2), the type becomes $[10 * 10; 5^6, 4^3, 3]$.

In the case (b), we get $Y' = 3(\nu_1 - 1) + 2$ and $Y' = s\nu_1 + 5$. Hence, $s\nu_1 + 5 = 3\nu_1 - 1$. In other words, $6 = (3 - s)\nu_1$. Thus, $s = 2$ and $\nu_1 = 6$. The type becomes $[12 * 12; 6^6, 5^3, 2]$.

If $\varepsilon(t) \geq 5$, then we find j such that $2\nu_1 - 4 \geq 2j(\nu_1 - j)$. Then $j^2 - 2 \geq (j - 1)\nu_1 \geq 2j^2 - 2j$. Hence,

$$0 \geq j^2 - 2j + 2.$$

Thus, $j < 2$, a contradiction.

(iv) Assume that $t_{\nu_1 - 1} = 4$.

If $\varepsilon(t) = 5$, then we find j such that $\widetilde{\mathcal{Z}} - 4(\nu_1 - 1) = \nu_1 - 3$. In this case, it is easy to derive a contradiction.

(v) Assume that $t_{\nu_1 - 1} = 0, t_{\nu_1 - 2} = 1$. Then

$$\widetilde{\mathcal{Z}} - 2(\nu_1 - 2) = 3\nu_1 - 3.$$

If $\varepsilon(t) = 2$, then we find j such that $3\nu_1 - 3 = j(\nu_1 - j)$. Thus

$$j + 3 + \frac{6}{j - 3} = \nu_1.$$

From this we obtain the next table: By $Y' = s\nu_1 + 5$, we conclude that this case does not happen.

Table 32

$j-3$	j	$j+3$	ν_1	$\nu_1 - j$	$\nu_1 - 2$	Y'
6	9	12	13	4	11	15
3	6	9	11	5	9	14
2	5	8	11	6	9	15
1	4	7	13	9	11	20

If $\varepsilon(t) \geq 3$, then we find j such that $3\nu_1 - 3 \geq 2j(\nu_1 - j)$. Thus

$$2j^2 - 3 \geq (2j-3)\nu_1 \geq 4j^2 - 6j.$$

Then $j < 3$.

(vi) Assume that $t_{\nu_1 - 1} = 0, t_{\nu_1 - 2} = 2$. Then

$$\widetilde{\mathcal{Z}} - 4(\nu_1 - 2) = \nu_1 + 1.$$

If $\varepsilon(t) = 3$, then we find j such that $\nu_1 + 1 = j(\nu_1 - j)$. Thus

$$j + 1 + \frac{2}{j-1} = \nu_1.$$

14.8.4. *case in which* $\omega = 4$ *and* $g = 1$.
(2) $\overline{g} = 0$. Then $\omega_1 = 4$ and

$$\widetilde{\mathcal{Z}} = 4\nu_1 - 4.$$

(i) First assume that $t_{\nu_1 - 1} = 1$.
If $\varepsilon(t) = 2$, then we find j such that $\widetilde{\mathcal{Z}} - (\nu_1 - 1) = 3\nu_1 - 3 = j(\nu_1 - j)$. Thus

$$j + 3 + \frac{6}{j-3} = \nu_1.$$

From this we obtain the next table: By $Y' = s\nu_1 + 4$, we conclude that

Table 33

$j-3$	j	$j+3$	ν_1	$\nu_1 - j$	$\nu_1 - 1$	Y'
6	9	12	13	4	12	16
3	6	9	11	5	10	15
2	5	8	11	6	10	16
1	4	7	13	9	12	21

$s = 1$ and $\nu_1 = 11$. Hence, the type becomes $[22 * 22; 11^7, 10, 5]$.
If $\varepsilon(t) \geq 3$, then we find j such that $3\nu_1 - 3 \geq 2j(\nu_1 - j)$. Then $j = 2$. In this case,

$$\widetilde{\mathcal{Z}} - (\nu_1 - 1) - 2(\nu_1 - 2) = \nu_1 + 1 \geq 2(\nu_1 - 2).$$

Hence, $\nu_1 = 4, 5$.
If $\nu_1 = 5$ then $s = 1$ and the type is $[10 * 10; 5^7, 4, 3, 2]$.

If $\nu_1 = 4$ then $s = 1$ and then $\widetilde{\mathcal{Z}} - (\nu_1 - 1) - 2(\nu_1 - 2) = \nu_1 + 1 = 5 = 4t_2$, a contradiction.

(ii) Assume that $t_{\nu_1 - 1} = 2$.

If $\varepsilon(t) = 3$, then we find j such that $\widetilde{\mathcal{Z}} - 2(\nu_1 - 1) = 2\nu_1 - 2 = j(\nu_1 - j)$. Thus $j^2 - 2 = (j-1)\nu_1$. Hence, if $j > 2$ then

$$j + 2 + \frac{2}{j-2} = \nu_1.$$

From this we obtain the next table: By $Y' = 7s + 4$, we arrive at a

TABLE 34

$j-2$	j	$j+2$	ν_1	$\nu_1 - j$	$\nu_1 - 1$	Y'
2	4	6	7	3	6	9
1	3	5	7	4	6	10

contradiction.

(iii) Assume that $t_{\nu_1 - 1} = 3$.

If $\varepsilon(t) = 4$, then we find j such that $\widetilde{\mathcal{Z}} - 3(\nu_1 - 1) = \nu_1 - 1 = j(\nu_1 - j)$. Thus $j^2 - 1 = (j-1)\nu_1$. Hence, $j = 1$ or $j = \nu_1 - 1$.

(iv) Assume that $t_{\nu_1 - 1} = 4$.

If $\varepsilon(t) = 4$, then we find j such that $\widetilde{\mathcal{Z}} - 4(\nu_1 - 1) = 0$. Thus $Y' = 4(\nu_1 - 1)$. Hence, $Y' = s\nu_1 + 4 = 4(\nu_1 - 1)$. Therefore, $8 = (4 - s)\nu_1$, which implies that 1) $\nu_1 = 8, s = 3$ or 2)$\nu_1 = 4, s = 2$.

In case 1), the type becomes $[16 * 16; 8^5, 7^4]$.

While,in case 2), the type becomes $[8 * 8; 4^6, 3^4]$.

(v) Assume that $t_{\nu_1 - 1} = 0, t_{\nu_1 - 2} = 1$.

If $\varepsilon(t) = 2$, then we find $j > 2$ such that $\widetilde{\mathcal{Z}} - 2(\nu_1 - 2) = 2\nu_1 = j(\nu_1 - j)$. Thus $j^2 - 4 + 4 = (j-1)\nu_1$. Hence,

$$j + 2 + \frac{4}{j-2} = \nu_1.$$

From this we obtain the next table: By $Y' = s\nu_1 + 4$, we conclude that

TABLE 35

$j-2$	j	$j+2$	ν_1	$\nu_1 - j$	$\nu_1 - 2$	Y'
4	6	8	9	3	7	10
2	4	6	8	4	6	10
1	3	5	9	6	7	13

$s = 1, \nu_9, j = 3$. The type becomes $[18 * 18; 9^7, 7, 6]$.

If $\varepsilon(t) \geq 3$, then we find $j > 2$ such that $\widetilde{\mathcal{Z}} - 2(\nu_1 - 2) = 2\nu_1 \geq 2j(\nu_1 - j)$.

(vi) Assume that $t_{\nu_1 - 1} = 0, t_{\nu_1 - 2} = 2$.

If $\varepsilon(t) = 2$, then we find $j > 2$ such that $\widetilde{\mathcal{Z}} - 2 \cdot 2(\nu_1 - 2) = 4 \geq 3(\nu_1 - 3)$. Hence, $\nu_1 = 4$. Then $t_3 = 0, t_2 = 2$. Thus, $Y' = 4$. But $Y' = 4s + 4 \geq 8$, a contradiction.

14.8.5. *case in which* $\omega = 4$ *and* $g > 0$.

(3) $\overline{g} = 1$. Then $\omega_1 = 3$ and

$$\widetilde{\mathcal{Z}} = 3\nu_1 - 1.$$

(i) First assume that $t_{\nu_1 - 1} = 1$.

If $\varepsilon(t) = 2$, then we find j such that $\widetilde{\mathcal{Z}} - (\nu_1 - 1) = 2\nu_1 = j(\nu_1 - j)$. Thus $j > 2$ and then

$$j + 2 + \frac{4}{j - 2} = \nu_1.$$

From this we obtain the next table: By $Y' = s\nu_1 + 3$, we conclude that

TABLE 36

$j-2$	j	$j+2$	ν_1	$\nu_1 - j$	$\nu_1 - 1$	Y'
4	6	8	9	3	8	11
2	4	6	8	4	7	11
1	3	5	9	6	8	14

$s = 1$ and $\nu_1 = 8$. Hence, the type becomes $[16 * 16; 8^7, 7, 4]$.

If $\varepsilon(t) \geq 3$, then we find j such that $2\nu_1 \geq 2j(\nu_1 - j)$. Then $j < 2$, a contradiction.

(ii) Assume that $t_{\nu_1 - 1} = 2$.

If $\varepsilon(t) = 3$, then we find j such that $\widetilde{\mathcal{Z}} - 2(\nu_1 - 1) = \nu_1 + 1 = j(\nu_1 - j)$. Thus if $j > 2$ then

$$j + 1 + \frac{2}{j - 1} = \nu_1.$$

From this we obtain the next table: Then $Y' = s\nu_1 + 3 = 5s + 3 \neq 10, 11$,

TABLE 37

$j-1$	j	$j+1$	ν_1	$\nu_1 - j$	$\nu_1 - 1$	Y'
2	3	4	5	2	4	10
1	2	3	5	3	4	11

a contradiction.

If $\varepsilon(t) > 3$, then we find j such that $\nu_1 + 1 \geq 2j(\nu_1 - j)$, a contradiction.

(iii) Assume that $t_{\nu_1-1}=3$.

If $\varepsilon(t)=4$, then we find j such that $\widetilde{\mathcal{Z}}-3(\nu_1-1)=2=j(\nu_1-j)$. This case does not occur.

(iv) Assume that $t_{\nu_1-1}=0, t_{\nu_1-2}=1$.

If $\varepsilon(t)=2$, then we find j such that $\widetilde{\mathcal{Z}}-2(\nu_1-2)=\nu_1+3=j(\nu_1-j)$. Thus

$$j+1+\frac{4}{j-1}=\nu_1.$$

From this we obtain the next table: Then $Y'=s\nu_1+3$; hence, $\nu_1=7$

TABLE 38

$j-1$	j	$j+1$	ν_1	ν_1-j	ν_1-2	Y'
4	5	6	7	2	5	7
2	3	4	6	3	4	7
1	2	3	7	5	5	10

and $j=2$.

The type becomes $[14*14;7^7,5^2]$.

If $\varepsilon(t)>2$, then we find j such that $\nu_1+3\geq 2j(\nu_1-j)$. Hence, $j<2$, a contradiction.

(4) $\overline{g}=2$. Then $\omega_1=2$ and

$$\widetilde{\mathcal{Z}}=2\nu_1+2.$$

(i) First assume that $t_{\nu_1-1}=1$.

If $\varepsilon(t)=2$, then we find j such that $\widetilde{\mathcal{Z}}-(\nu_1-1)=\nu_1+3=j(\nu_1-j)$. Thus

$$j+1+\frac{4}{j-1}=\nu_1.$$

From this we obtain the next table: By $Y'=s\nu_1+2$, we conclude that

TABLE 39

$j-2$	j	$j+2$	ν_1	ν_1-j	ν_1-1	Y'
4	5	6	7	2	6	8
2	3	4	6	3	5	8
1	2	3	7	5	6	11

$s=1$ and $\nu_1=6$. Hence, the type becomes $[12*12;6^7,5,3]$.

(ii) Assume that $t_{\nu_1-1}=2$.

If $\varepsilon(t)\geq 2$, then we find j such that $\widetilde{\mathcal{Z}}-2(\nu_1-1)=4\geq 2\nu_1-4$. Thus $\nu_1=4$.

Hence, $\widetilde{\mathcal{Z}}=10=3t_3+4t_2$. Thus, $t_3=2, t_2=1$ and so $Y'=6+2=8$. Since $Y'=4s+2$, we derive a contradiction.

(5) $\overline{g} = 3$. Then $\omega_1 = 1$ and

$$\widetilde{\mathcal{Z}} = \nu_1 + 5.$$

(i) First assume that $t_{\nu_1 - 1} = 1$.

If $\varepsilon(t) \geq 2$, then we find j such that $\widetilde{\mathcal{Z}} - (\nu_1 - 1) = 6 \geq 2(\nu_1 - 4)$. Thus $\nu_1 = 4$ or 5. Moreover, in the other cases, $\nu_1 \leq 5$ is verified.

Therefore, we have two cases:

(i) $\nu_1 = 4$. Then

$$\widetilde{\mathcal{Z}} = \nu_1 + 5 = 9 = 3t_3 + 4t_2.$$

We have a solution:$t_3 = 3, t_2 = 0$. Hence, $Y' = 9$ and $Y' = 4s + 1 = 9$. Therefore, $s = 2$ and the type becomes $[8 * 8; 4^6, 3^3]$.

(ii) $\nu_1 = 5$. Then

$$\widetilde{\mathcal{Z}} = \nu_1 + 5 = 10 = 4t_4 + 6(t_3 + t_2).$$

$t_4 = 1, t_3 + t_2 = 1$. Hence, $Y' = 6$ or 7. By $Y' = 5s + 1$, we conclude that $s = 1$ and $t_2 = 1$. Therefore, the type becomes $[10 * 10; 5^7, 4, 2]$.

(6) $\overline{g} = 4$. Then $\omega_1 = 0$ and

$$\widetilde{\mathcal{Z}} = 8.$$

From this we obtain the next table: By $Y' = s\nu_1 + 2$, we conclude that

Table 40

$j-2$	j	$j+2$	ν_1	$\nu_1 - j$	$\nu_1 - 1$	Y'
4	5	6	7	2	6	8
2	3	4	6	3	5	8
1	2	3	7	5	6	11

$s = 1$ and $\nu_1 = 6$. Hence, the type becomes $[12 * 12; 6^7, 5, 3]$.

15. sharp estimate

Here we shall show the following result. Suppose that $\sigma \geq 7$.

Theorem 8. (1) $\sigma \leq (\omega + 1)(\omega + 2)$.

(2) *If* $\sigma = (\omega + 1)(\omega + 2)$ *then the type is* $[2\nu_1 * 2\nu_1; \nu_1^7, \nu_1 - 1, \nu_r]$, *where* $\nu_1 = \frac{\nu_r(\nu_r - 1)}{2}$ *and* $\omega = \nu_r - 2$.

(3) *If* $\sigma < (\omega + 1)(\omega + 2)$ *then* $\sigma \leq \omega(\omega + 1) + 2$ *except for the following cases;*

(a) $(\omega = 2)$, $[10 * 11; 5^9]$;

(b) $(\omega = 3)$, $[15 * 22, 1; 7^9]$ *and* $[16 * 16; 8^6, 7^2, 6]$;

(4) *If* $\sigma < \omega(\omega + 1) + 2$ *then* $\sigma \leq \omega(\omega - 1) + 4$ *except for the following cases;*

(a) $(\omega = 3, g = 1)$, $[12 * 12; 6^6, 5]$;

(b) $(\omega = 4, g = 1)$, $[18 * 18; 9^7, 7, 6]$;
(c) $(\omega = 4, g = 0)$, $[19 * 19; 9^9]$;
(d) $(\omega = 4g = 0)$, $[20 * 20; 10^5, 9^3, 8]$.

Theorem 9. (1) $\sigma \leq (\alpha + 3)(\alpha + 2)$ *(By O.Matsuda)*;

(2) *If* $\sigma = (\alpha+3)(\alpha+2)$ *then the type is* $[2\nu_1 * 2\nu_1; \nu_1^7, \nu_1 - 1, \nu_r]$, *where* $\nu_1 = \frac{\nu_r(\nu_r-1)}{2}$ *and* $\omega = \nu_r - 2$.

(3) $\sigma \leq \alpha^2 + (1 - 4\overline{g})\alpha + 4\overline{g}^2 + 2$;

(4) *If* $g > 0$ *then* $\sigma \leq \alpha(\alpha + 1) + 2$;

(5) *If* $\sigma < (\alpha + 3)(\alpha + 2)$ *then* $\sigma \leq \alpha(\alpha + 1) + 2$,*except for the following cases;*

(a) $(\alpha = 1)$, $[10 * 11; 5^9]$;
(b) $(\alpha = 2)$, $[15 * 22, 1; 7^9]$, $[16 * 16; 8^6, 7^2, 6]$;
(c) $(\alpha = 3)$, $[19 * 19; 9^9]$, $[19 * 38, 2; 9^9]$, $[20 * 20; 10^5, 9^3, 8]$, $[22 * 22; 11^6, 10^2, 7]$, $[22 * 22; 11^7, 8^2]$;
(d) $(\alpha = 4)$, $[23 * 35, 1; 11^9]$, $[24 * 24; 12^4, 11^4, 10]$, $[24 * 25; 12^7, 10^2]$, $[25 * 37, 1; 12^8, 9]$, $[28 * 29; 14^8, 8]$, $[30 * 30; 15^7, 13, 8]$;
(e) $(\alpha = 5)$, $[36 * 37; 18^8, 9]$, $[38 * 38; 19^7, 17, 9]$;
(f) $(\alpha = 6)$, $[46 * 46; 23^6, 22^2, 10]$.

Proof. By Theorem 6, we can assume that $g = 0$. Hence, $\omega_1 = \omega + 1$. In particular, $D^2 < 0$.

15.1. **case in which $B \geq 3$.**

(1) If $B \geq 3$ and $D^2 \leq 0$, then $\sigma \leq \frac{4\omega}{3} + 3$. Hence, by $\sigma \geq 8$, we get $8 \leq \frac{4\omega}{3} + 3$ and so $\omega \geq \frac{15}{4}$; hence $\omega \geq 4$. Further, by

$$
\begin{aligned}
(\omega - 1)\omega - (\frac{4\omega}{3} + 3) &= \frac{\omega(3\omega - 7)}{3} - 3 \\
&\geq \frac{4(3 \times 4 - 7)}{3} \\
&\geq 2
\end{aligned}
$$

we obtain

$$
(\omega - 1)\omega > \sigma.
$$

(2) We assume that $B \leq 2$. By the fundamental equalities, we obtain

$$
\tilde{\mathcal{Z}} = -\lambda\nu_1 - \omega_1 + 2\overline{g} - \tilde{k} \geq 0.
$$

Hence, thanks to $g = 0$, we get

$$
\lambda\nu_1 \leq -\omega_1 + 2\overline{g} - \tilde{k} \leq -\omega_1 - 3 \leq -6.
$$

Thus

$$
\lambda = k - \omega_1 < 0.
$$

15.2. **case in which $\lambda < 0$ and $p \geq 1$.**
First, we recall $\lambda < 0$.
If $p \geq 1$, then recalling the inequality (15), we get

$$\sigma \leq \frac{\omega_1^2 + 9\omega_1 - 2}{3}. \tag{30}$$

We wish to prove the inequality

$$\omega(\omega - 1) + 2 = (\omega_1 - 1)(\omega_1 - 2) + 2 \geq \sigma.$$

Then

$$(\omega_1 - 1)(\omega_1 - 2) + 2 - (\frac{\omega_1^2 + 9\omega_1 - 2}{3}) = \frac{2\omega_1(\omega_1 - 9) + 14}{3}.$$

If $\omega_1 = 8$, then the right hand side turns out to be $-\frac{2}{3}$. Thus, when $\omega_1 \geq 8$ the inequality is verified.
To study the case when $\omega_1 < 8$, we recall the fundamental equalities:

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $Y = 8\nu_1 + k + \omega_1$.

Here $\tilde{k} = p(k - 2p)$.
By using Lemma of Tanaka and Matsuda, we get

$$V = 8\nu_1^2 + (k + \omega_1)^2 - X \geq 0.$$

Thus

$$k + 2\omega_1 + \frac{\omega_1(\omega_1 - 1) - \tilde{k} + pk - 2}{k} \geq \sigma,$$

and so

$$k + 2\omega_1 + \frac{\omega_1(\omega_1 - 1) + 2p^2 - 2}{k} \geq \sigma,$$

The integral part of the left hand side is denoted by $W(\omega_1, k)$.
In that follows, we shall check the inequality by distinguishing the following cases according to the value of $\omega_1 \leq 7$.

15.2.1. *case in which $\omega_1 = 7$.*
By $\lambda = k - \omega_1 = k - 7 < 0$, we have $k \leq 6$.
It is easy to see that if $p = 1$, then $W(7,3) = 31, W(7,4) = 28, W(7,5) = 27, W(7,6) = 27$.
If $p = 2$, then $W(7,6) = 28, W(7,8) = 28$.
But $\omega(\omega - 1) + 2 = 32$. Hence,

$$\sigma \leq W(7,k) \leq 32 = \omega(\omega - 1) + 2.$$

15.2.2. *case in which $\omega_1 = 6$.*
By $\lambda = k - \omega_1 = k - 6 < 0$, we have $k \leq 5$. Then $p = 1$ and

$$W(6,3) = 25, W(6,4) = 23, W(6,5) = 23.$$

We distinguish the following cases according to the value of k.
(1) $k = 3$. Then $p = 1, w = 3, u = 0$. Then by the fundamental equalities

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega - 3\overline{g} = 8\nu_1^2 + 6\nu_1 + 1 + 5 + 3$,

- $Y = 8\nu_1 + k + \omega_1 = 8\nu_1 + 9$,

we get

$$\widetilde{\mathcal{Z}} = 3\nu_1 - 9 \geq (3\nu_1 - 9)t_{\nu_1 - 3}.$$

Thus if $t_{\nu_1-3} = 1$ then Y becomes $t\nu_1 + (\nu_1 - 3) = (t+1)\nu_1 - 3$. Combining this with $Y = 8\nu_1 + 9$, we get

$$12 = (t - 7)\nu_1.$$

We have two solutions.

- $\nu_1 = 12, t = 8$. The type becomes $[25 * 37, 1; 12^8, 9]$;
- $\nu_1 = 7, t = 8$. The type becomes $[13 * 19, 1; 6^9, 3]$.

But if $t_{\nu_1-3} = 0$ then $\widetilde{\mathcal{Z}} = \nu_1 - 1$ or $= 2(\nu_1 - 1)$ or $= 2\nu_1 - 4$. In these cases, $\nu_1 \leq 7$.

If $\nu_1 = 7$ then $\widetilde{\mathcal{Z}} = 3(\nu_1 - 3) = 12$; hence, $12 = 6t_6 + 10t_5 + 12t_4 + \cdots$. $t = 6 = 2$; and $Y = \nu_1 t + 12 = 8\nu_1 + 9$. Then $3 = (8 - t)\nu_1 = (8 - t)7$; a contradiction.

If $\nu_1 = 6$ then $\widetilde{\mathcal{Z}} = 3(\nu_1 - 3) = 9$; hence, $9 = 5t_5 + 8t_4 + 9t_3 + \cdots$, which has no solution.

If $\nu_1 = 5$ then $\widetilde{\mathcal{Z}} = 6$; hence, $6 = 4t_4 + 6(t_3 + t_2)$. Then $t_3 + t_2 = 1$. By $Y = 5t + 3$ or $5t + 2$. From both we can derive contradictions.

If $\nu_1 = 4$ then $\widetilde{\mathcal{Z}} = 3$; hence, $3 = 3t_3 + 4t_2$. Then $t_3 = 1$. By $Y = 4t + 3$, we get $Y = 8\nu_1 + 9 = 4t + 3$, a contradiction.

(2) $k = 4$. Then $p = 1, w = 4, u = 0$. By the fundamental equalities

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g} = 8\nu_1^2 + 8\nu_1 + 2 + 5 + 3$,
- $Y = 8\nu_1 + k + \omega_1 = 8\nu_1 + 10$,

we get

$$\widetilde{\mathcal{Z}} = 2\nu_1 - 10.$$

Thus $\nu_1 = 5$ and Y becomes $t\nu_1 = 8\nu_1 + 10 = 10\nu_1$. Hence, $t = 9$ and The type becomes $[10 * 10; 5^{10}]$.

(3) $k = 5$. Then $p = 1, w = 3, u = 1$. Then by the fundamental equalities

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega - 3\overline{g} = 8\nu_1^2 + 10\nu_1 + 3 + 5 + 3$,
- $Y = 8\nu_1 + k + \omega_1 = 8\nu_1 + 11$,

we get

$$\widetilde{\mathcal{Z}} = \nu_1 - 11.$$

Thus $\nu_1 = 11$ and Y becomes $t\nu_1 = 8\nu_1 + 11 = 9\nu_1$. Hence, $t = 9$ and the type becomes $[23 * 35, 1; 11^9]$.

15.2.3. *case in which* $\omega_1 \leq 5$.

In this case, $\omega = \omega_1 - 1 \leq 4$. These cases have already been treated in the former section.

15.3. case in which $\lambda < 0$ and $p = 0, u > 0$.

If $\lambda = k - \omega_1 < 0$ and $p = 0, u > 0$ then $k = 2u$.

The fundamental equalities become

- $X = 8\nu_1^2 + 2k\nu_1 + \omega_1 + 2$,
- $Y = 8\nu_1 + k + \omega_1$.

By Lemma of T. and M, we obtain

$$V = (k + \omega_1)^2 - 2k\nu_1 - \omega_1 - 2 \geq 0,$$

and

$$k + 2\omega_1 + \frac{\omega_1^2 - \omega_1 - 2}{k} \geq \sigma. \tag{31}$$

Further,

$$\begin{aligned}(\omega_1 - 1)(\omega_1 - 2) + 2 - (k + 2\omega_1 + \frac{\omega_1^2 - \omega_1 - 2}{k})\\ = \omega_1((1 - \frac{1}{k})\omega_1 - 3 + \frac{1}{k}) + 2 + \frac{2}{k} - k,\end{aligned}$$

which is written as $F(\omega_1)$. As a quadratic function $F(\omega_1)$ is increasing for $\omega_1 > 4$.

$$F(8) = 28 - k - \frac{54}{k} > 0 \text{ except for } k = 2.$$

But for $\omega_1 = 8, k = 2$, we get

$$k + 2\omega_1 + \frac{\omega_1^2 - \omega_1 - 2}{k} = 45 \geq \sigma.$$

Since $\sigma = 2\nu_1$, it follows that $\nu_1 \leq 22$.

Noting that $(\omega_1 - 1)(\omega_1 - 2) + 2 = 44$ for $\omega_1 = 8$, we conclude that for $\omega_1 \geq 8$

$$(\omega_1 - 1)(\omega_1 - 2) + 2 \leq \sigma.$$

15.3.1. *case in which $\omega_1 = 7$.*

Then we get

$$\tilde{\mathcal{Z}} = -2u\nu_1 + 7(\nu_1 - 1) - 2 = (7 - 2u)\nu_1 - 9.$$

Hence, $u \leq 3$.

(i) $u = 3$. Then

$$\tilde{\mathcal{Z}} = \nu_1 - 9.$$

Thus $\nu_1 = 9$ and Y becomes $t\nu_1 = 8\nu_1 + 6 + 7$, a contradiction.

(ii) $u = 2$. Then

$$\tilde{\mathcal{Z}} = 3(\nu_1 - 3).$$

Thus (a) either $t_{\nu_1 - 3} = 1$ or (b) $\tilde{\mathcal{Z}} = (\nu_1 - 1)t_{\nu_1 - 1} + (2\nu_1 - 4)t_{\nu_1 - 2}$.

In case (a), Y becomes $t\nu_1 + \nu_1 - 3 = 8\nu_1 + 4 + 7$. Hence, $(t - 7)\nu_1 = 14$. Therefore, we have the following three cases:

- $\nu_1 = 14, t = 8$; the type becomes $[28 * 30; 14^8, 11]$.
- $\nu_1 = 7, t = 9$; the type becomes $[14 * 16; 7^9, 4]$.

In case (b), if $t_{\nu_1-1} = 1, t_{\nu_1-2} = 0$ then $\nu_1 = 4$. $\widetilde{\mathcal{Z}} = 3$ and so $t_3 = 3$ and $Y = t\nu_1 + 9$. But $Y = 8\nu_1 + 9$. Hence, $t = 8$ and the type is $[8 * 9; 4^8, 3^3]$.

If $t_{\nu_1-1} = 2, t_{\nu_1-2} = 0$ then $\nu_1 = 7$. $\widetilde{\mathcal{Z}} = 12 = 6t_6 + 10(t_5 + t_2) + 12(t_4 + t_3)$.

(iii) $u = 1$. Then

$$\widetilde{\mathcal{Z}} = 5\nu_1 - 9.$$

If $t_{\nu_1-1} = 0$ then assume $\varepsilon(t) = 1$; we will find j such that $5\nu_1 - 9 = j(\nu_1 - j)$. Hence,

$$j + 5 + \frac{16}{j-5} = \nu_1.$$

Then we have the next table.

TABLE 41

	$j-5$	j	$j+5$	ν_1
1	16	21	26	27
2	8	13	18	20
4	4	9	14	18
8	2	7	12	20
16	1	6	11	27

Then $Y = t\nu_1 + j = 8\nu_1 + 9$; hence, $j = 9$ and $t = 8$. The type becomes $[36 * 37; 18^8, 9]$.

If $t_{\nu_1-1} > 1$ then

$$\widetilde{\mathcal{Z}} - (\nu_1 - 1) = 4(\nu_1 - 2).$$

Thus $t_{\nu_1-2} = 2$ and Y becomes $t\nu_1 + \nu_1 - 1 + 2(\nu_1 - 2) = 8\nu_1 + 9$. Hence, $(t-5)\nu_1 = 14$. Therefore, we have the following three cases:

- $\nu_1 = 14, t = 6$; the type becomes $[28 * 29; 14^6, 13, 12^2]$.
- $\nu_1 = 7, t = 7$; the type becomes $[14 * 15; 7^7, 6, 5^2]$.

15.3.2. *case in which* $\omega = 5$. Then by

- $X' = s\nu_1^2 + 8$,
- $Y' = s\nu_1 + 6$,

we get

$$\widetilde{\mathcal{Z}} = 6\nu_1 - 8.$$

By

$$\sigma \le s + 2\omega_1 + 1 + \frac{\omega_1^2 - \omega_1 + 2\overline{g}}{s} = s + 13 + \frac{28}{s}, \tag{32}$$

we get $\sigma = 2\nu_1 \le 28$.

What we wish to prove is the next inequality:

$$\sigma \le \omega^2 - \omega + 2 = 22.$$

Hence we shall study under the hypothesis $\nu_1 = 12, 13, 14$.

case in which $\nu_1 = 14$

TABLE 42. $\nu_1 = 14$

ω_1	ν_1	$\nu_1 - 1$	0	0	0	0	0	
6	14	13	12	11	10	9	8	7
0	0	1	2	3	4	5	6	7
0	76	13	24	33	40	45	48	49
0	0	63	52	43	36	31	28	27

The equation

$$\tilde{Z} = 76 = 13x_0 + 24x_1 + 33x_2 + 40x_3 + 45x_4 + 48x_5 + 49x_6$$

has no solution.

case in which $\nu_1 = 13$

TABLE 43. $\nu_1 = 13$

ω_1	ν_1	$\nu_1 - 1$	0	0	0	0	
6	13	12	11	10	9	8	7
0	0	1	2	3	4	5	6
0	70	12	22	30	36	40	42
0	0	58	48	40	34	30	28

The equation

$$\tilde{Z} = 70 = 12x_0 + 22x_1 + 30x_2 + 36x_3 + 40x_4 + 42x_5$$

has a solution $x_0 = 1, x_1 = 1, x_3 = 1$. Then $t_{12} = 1, t_{11} + t_2 = 1, t_9 + t_4 = 1$. But $Y' = 12 + 11 + 9 = 32, Y' = 13s + 6$. Hence, $s = 2$ and the type is $[26 * 26; 13^6, 12, 11, 9]$.

case in which $\nu_1 = 12$

TABLE 44. $\nu_1 = 12$

ω_1	ν_1	$\nu_1 - 1$	$\nu_1 - 1$	0	0	0	
6	12	11	10	9	8	7	6
0	0	1	2	3	4	5	6
0	64	11	20	27	32	35	36
0	0	53	44	37	32	29	28

The equation

$$\tilde{Z} = 64 = 11x_0 + 20x_1 + 27x_2 + 32x_3 + 35x_4 + 36x_5$$

has a solution $x_0 = 4, x_1 = 1$.

But $Y' = 44 + 10, Y' = 12s + 6$. Hence, $s = 4$ and the type is $[24 * 24; 12^4, 11^4, 10]$.

In the case when $\omega \leq 4$, we have already done it.

15.4. **case in which $\lambda < 0$ and $k = 0$.**

(5) Supposing $k = 0$ and $\lambda = -\omega_1 < 0$, we shall verify the result.

We distinguish the following cases according to the value of $t_{\nu_1 - 1}$.

15.5. **case in which $t_{\nu_1-1} = 0$.**

$(5-1)$ $t_{\nu_1-1} = 0$.

By the fundamental equalities, we get

- $Y' = s\nu_1 + \omega_1 = s\nu_1 + \omega + 1 = s(\nu_1 - 2) + 2s + \omega + 1$,
- $X' = s\nu_1^2 + \omega_1 - 2\overline{g} = s\nu_1^2 + \omega + 3$.

Then

$$\widetilde{\mathcal{Z}} = \nu_1(\omega + 1) - \omega - 3 \tag{33}$$

and by $Y' = s\nu_1 + \omega_1 = s\nu_1 + \omega + 1$, there exist at least $s + 1$ multiplicities ν_j with $\nu_j < \nu_1$. Hence,

$$\widetilde{\mathcal{Z}} = \nu_1(\omega + 1) - \omega - 3 \geq 2(s + 1)(\nu_1 - 2). \tag{34}$$

Lemma of Tanaka and Matsuda implies

$$V = s(\nu_1 - 2)^2 + (2s + \omega + 1)^2 - (s\nu_1^2 + \omega + 3) \geq 0.$$

By $\sigma = 2\nu_1$, we obtain

$$2s + 2\omega + 4 + \frac{\omega^2 + \omega - 2}{2s} \geq \sigma. \tag{35}$$

The following inequality is what we have to prove.

$$\omega(\omega - 1) + 2 \geq \sigma. \tag{36}$$

Subtracting the left hand side of (31) from that of (30), we get

$$\omega(\omega - 1) + 2 - (2s + 2\omega + 4 + \frac{\omega^2 + \omega - 2}{2s}),$$

which is written as $F(\omega)$. We shall show that $F(\omega) > 0$ for $\omega \geq 8$. Indeed, Then $F(8) \geq 1$ for $1 \geq s \geq 7$.

But $F(7) = -3$ for $s = 1$.

TABLE 45

ν_1	$\nu_1 - 1$										
23	22	21	20	19	18	17	16	15	14	13	12
	1	2	3	4	5	6	7	8	9	10	11
	22	42	60	76	90	102	112	120	126	130	132
		132	114	98	84	72	62	54	48	44	42

15.5.1. *case in which* $\omega = 7$.

Assume $\omega = 7$. Then, $\omega_1 = 8$.

When $\nu_1 = 23$, we have $\tilde{Z} = 8\nu_1 - 1 = 173$ and the following table.

The next equation

$$\tilde{Z} = 174 = 42x_1 + 60x_2 + 76x_3 + 90x_4 + 102x_5 + 112x_6 + 120x_7 + 126x_8 + 130x_9 + 132x_{10}$$

has no solution.

15.5.2. *case in which* $\omega = 6$.

15.5.3. *case in which* $\omega = 5$.

Put $x_1 = t_{\nu_1-2} + t_2, x_2 = t_{\nu_1-3} + t_3, \cdots$.

In general, if $\nu > 2j$ then put $x_{j-1} = t_{\nu_1-j} + t_j$, and if $\nu = 2j$ then $x_{j-1} = t_j$.

Moreover, put $x_0 = t_{\nu_1-1}$.

Then we have

$$\tilde{\tilde{Z}} = 6\nu_1 - 8 = (\nu_1 - 1)x_0 + 2(\nu_1 - 2)x_1 + 3(\nu_1 - 3)x_3 + \cdots .$$

Under the hypothesis $x_0 = t_{\nu_1-1} = 0$, we shall distinguish the various cases:

(i) $x_1 = 1$. Then

$$\tilde{\tilde{Z}}' = \tilde{\tilde{Z}} - 2(\nu_1 - 2) = 4(\nu_1 - 1).$$

(a)

Suppose that $\varepsilon(t) = 2$. Then for some $j > 2$, $4(\nu_1 - 1) = j(\nu_1 - j)$.

$$j^2 - 16 + 12 = (j - 4)\nu_1,$$

and so

$$j + 4 + \frac{12}{j - 4} = \nu_1$$

From this we obtain the next table:

Therefore, $x_1 = t_{\nu_1-2} + t_2 = 1, x_j = t_{\nu_1-j} + t_j = 1$. Moreover, $Y' = s\nu_1 + 6$. Thus, we conclude that $s = 1$ and $\nu_1 = 15, t_{13} = t_7 = 1$.

Therefore, the type becomes $[30 * 30; 15^7, 13, 7]$.

Except for the above type, the next inequality is satisfied.

TABLE 46

$j-4$	j	$j+4$	ν_1	ν_1-1	ν_1-j
12	16	20	21	20	9
6	10	14	16	15	10
4	8	12	15	14	11
3	7	11	15	14	12
2	6	10	16	15	14
1	5	9	21	20	20

$$\omega(\omega-1)+2 \geq \sigma.$$

(b)
Suppose that $\varepsilon(t) \geq 3$. Then there exists $j > 2$ such that $4(\nu_1-1) \geq 2j(\nu_1-j)$.
Then
$$2(\nu_1-1) \geq j(\nu_1-j)$$
and so since $\nu_1 \geq 2j$, it follows that
$$j^2-2 \geq (j-2)\nu_1 \geq 2j^2-4j.$$
Hence,
$$0 \geq j^2-4j+2 = (j-2)^2-2.$$
Thus $j=3$. Therefore, may assume $x_2=1$. Hence,
$$\widetilde{\mathcal{Z}}' = \widetilde{\mathcal{Z}} - 2(\nu_1-2) - 3(\nu_1-3) = \nu_1+5.$$
Then there exists $j > 2$ such that $\nu_1+5 = j(\nu_1-j)$.
Hence,
$$j^2-1+6 = (j-1)\nu_1 \geq 2j(j-1).$$
Thus,
$$5 \geq j^2-2j = (j-1)^2-1.$$
Hence, $j=3$ which implies $x_2=2$. Furthermore, $\nu_1=7$. By $x_1=1, x_2=2$, we have $1=t_5+t_2, 2=t_4+t_3$. But $Y'=s\nu_1+6=7s+6$. Therefore, $t_5=1, t_4=2, s=1$. The type becomes $[14*14; 7^7, 5, 4^2]$.

(ii) $x_1=2$. Then

$$\widetilde{\mathcal{Z}}' = \widetilde{\mathcal{Z}} - 2\cdot 2(\nu_1-2) = 2\nu.$$

(a)
Suppose that $\varepsilon(t)=2$. Then for some $j>2$, $2\nu_1 = j(\nu_1-j)$.

$$j^2-4+4 = (j-2)\nu_1,$$

and so

$$j + 2 + \frac{4}{j-2} = \nu_1$$

From this we obtain the next table:

TABLE 47

$j-2$	j	$j+2$	ν_1	ν_1-1	ν_1-j
4	6	8	9	8	5
2	4	6	8	7	6
1	3	5	9	8	8

Therefore, if $\nu_1 = 9$ then $Y' = 9s + 6$. By $2 = t_7 + t_2 = 2, , t_6 = 1$, we get $t_7 = 1, t_2 = 1, t_6 = 1, s = 1$.

The type becomes $[18 * 18; 9^7, 7, 6, 2]$.

(b)

It is not hard to derive a contradiction from $\varepsilon(t) > 2$.

(iii) $x_1 \geq 3$. Then

$$\widetilde{\widetilde{\mathcal{Z}}}' = \widetilde{\widetilde{\mathcal{Z}}} - 3 \cdot 2(\nu_1 - 2) \leq 4.$$

Suppose that $\varepsilon(t) \geq 2$. Then for some $j > 2$, $4 \geq 3(\nu_1 - 3)$.

Hence, $\nu_1 = 4$.

In general, when $\nu_1 = 4$, one has $\widetilde{\widetilde{\mathcal{Z}}} = 16 = 3t_3 + 4t_2$. Thus, $t_2 = 4$ and $Y' = 8, Y' = 4s + 6$, a contradiction.

15.5.4. *case in which* $\omega \leq 4$.

This case has alreasdy been treated in the former sections.

15.6. **case in which $t_{\nu_1 - 1} > 0$ and $s \geq 2$.**

(5 − 2) $t_{\nu_1-1} > 0, s \geq 2$.

Then by the fundamental equalities, we get

- $Y' = s\nu_1 + \omega_1 = s\nu_1 + \omega_1 = s(\nu_1 - 1) + s + \omega_1$,
- $X' = s\nu_1^2 + \omega_1 - 2\overline{g}$.

Then Lemma of Tanaka and Matsuda implies

$$V = s(\nu_1 - 1)^2 + (s + \omega_1)^2 - (s\nu_1^2 + \omega_1 - 2\overline{g}) \geq 0.$$

By $\sigma = 2\nu_1$, we obtain

$$s + 2\omega_1 + 1 + \frac{\omega_1^2 - \omega_1 + 2\overline{g}}{s} \geq \sigma. \tag{37}$$

Recalling that $g = 0$, we get

$$s + 2\omega + 3 + \frac{\omega^2 + \omega - 2}{s} \geq \sigma. \tag{38}$$

The following inequality is what we have to prove.

$$\omega(\omega-1)+2 \geq \sigma. \tag{39}$$

Hence, defining $F(x)$ to be $x(x-1)+2-(s+2x+3+\frac{x^2+x-2}{s})$, we investigate the value $F(\omega)$.

Then $F(8) = 58-(s+19+\frac{70}{s}) > 0$ and
if $F(7) = 44-(s+17+\frac{54}{s}) < 0$ then $s=2$ and $s+17+\frac{54}{s} = 46$.

15.6.1. *case in which* $\omega = 7$. Assume $\omega = 7$. Then, $\omega_1 = 8$.

When $\nu_1 = 23$, we have $\tilde{Z} = 8\nu_1 - 1 = 173$ and the following table.

TABLE 48

ν_1	$\nu_1 - 1$										
23	22	21	20	19	18	17	16	15	14	13	12
	1	2	3	4	5	6	7	8	9	10	11
	22	42	60	76	90	102	112	120	126	130	132
		132	114	98	84	72	62	54	48	44	42

The next equation

$\tilde{Z} = 174 = 22x_0+42x_1+60x_2+76x_3+90x_4+102x_5+112x_6+120x_7+126x_8+130x_9+132x_{10}$

has solution $x_0 = 2, x_{12} = 1$. Hence, $t_{22} = 2$ and $t_{13}+t_{10} = 1$.

$Y' = 44+13$ or $44+10$. But $Y' = 23s+8$. Hence, $s=2$ and $Y' = 54$.

Thus the type becomes $[46*46; 23^6, 22^2, 10]$.

15.6.2. *case in which* $\omega = 6$.

Assume $\omega = 6$. Then, $\omega_1 = 7$. When $\nu_1 = 18$, we have $\tilde{Z} = 117$ and the following table.

TABLE 49

ν_1	$\nu_1 - 1$								
18	17	16	15	14	13	12	11	10	9
0	1	2	3	4	5	6	7	8	9
	17	32	45	56	65	72	77	80	81
	100	85	72	61	52	45	40	37	36

The equation

$\tilde{Z} = 117 = 17x_0+32x_1+45x_2+56x_3+65x_4+72x_5+77x_6+80x_7+81x_8$

has a solution $x_2 = x_5 = 1$. Hence, $t_{15}+t_3 = 1$ and $t_{12}+t_6 = 1$. Thus $Y' = 15+12$ or $15+6$ or $3+12$ or $3+6$. By $Y' = 18s+7$, we have a contradiction.

When $\nu_1 = 17$, we have $\tilde{Z} = 110$ and the following table.

TABLE 50. $\nu_1 = 17$

$\omega 1$	ν_1	$\nu_1 - 1$	$\nu_1 - 1$	$\nu_1 - 1$	$\nu_1 - 1$	$\nu_1 - 1$	$\nu_1 - 1$	$\nu_1 - 1$	$\nu_1 - 1$
7	17	16	15	14	13	12	11	10	9
	0	1	2	3	4	5	6	7	8
	110	16	30	42	52	60	66	70	72
		101	87	75	65	57	51	47	45

The equation

$$\tilde{Z} = 110 = 16x_0 + 30x_1 + 42x_2 + 52x_3 + 60x_4 + 66x_5 + 70x_6 + 82x_7$$

has no solution.

Therefore, when $\omega = 6$, we get $\sigma \leq \omega^2 - \omega + 2 = 32$ and $\nu_1 \leq 16$.

15.7. **case in which** $t_{\nu_1 - 1} > 0$ **and** $s = 1$.

(5 − 3)

Then $t = 8 - s = 7$ and if $r = t + 2 = 9$ then

- $Y' = \nu_1 - 1 + \nu_r$,
- $X' = (\nu_1 - 1)^2 + {\nu_r}^2$.

Thus,

- $Y' = s\nu_1 + \omega_1 = \nu_1 - 1 + \omega + 2$,
- $X' = s\nu_1^2 + \omega_1 - 2\overline{g} = \nu_1^2 + \omega + 3$.

Hence,

$$V = (\nu_1 - 1)^2 + (\omega + 2)^2 - X' = 0.$$

From this, $\nu_r = \omega + 2$ and by $g = 0$ we get $\nu_1 = \frac{\nu_r(\nu_r - 1)}{2}$ and so

$$\sigma = (\omega + 1)(\omega + 2),$$

which contradicts the hypothesis.

15.8. **case in which** $r = t + 2 > 9$.

Further, suppose that $r = t + 2 > 9$. Let the multiplicites of C be denoted by ν_1^t(, which means that there are t multiple points of multiplicity ν_1),$\nu_1 - 1, \varepsilon_1, \varepsilon_2, \cdots$. Then

- $Y' = \nu_1 - 1 + \varepsilon_1 + \varepsilon_2 + \cdots = s\nu_1 + \omega_1$,
- $X' = (\nu_1 - 1)^2 + \varepsilon_1^2 + \varepsilon_2^2 + \cdots = s\nu_1^2 + \omega_1 - 2\overline{g}$.

Putting

$$\varepsilon_2' = \varepsilon_2 + \cdots, \varepsilon_2'' = \varepsilon_2^2 + \cdots \geq {\varepsilon_2'}^2,$$

we get

- $Y' = \nu_1 - 1 + \varepsilon_1 + \varepsilon_2' = s\nu_1 + \omega_1$,
- $X' = (\nu_1 - 1)^2 + \varepsilon_1^2 + \varepsilon_2'' = s\nu_1^2 + \omega_1 - 2\overline{g}$.

Then since $s = 1$ and $\overline{g} = -1$, it follows that

- $\varepsilon_1 + \varepsilon_2' = \omega + 2$,

- $\varepsilon_1^2 + \varepsilon_2'' = 2\nu_1 + \omega + 2$.

and so

$$\begin{aligned} 2\varepsilon_1\varepsilon_2' &= (\varepsilon_1 + \varepsilon_2')^2 - \varepsilon_1^2 - {\varepsilon_2'}^2 \\ &\le (\varepsilon_1 + \varepsilon_2')^2 - \varepsilon_1^2 - \varepsilon_2'' \\ &\le (\omega + 2)^2 - (2\nu_1 + \omega + 2) \\ &= \omega^2 + 3\omega + 2 - \sigma. \end{aligned}$$

Hence,

$$2\varepsilon_1\varepsilon_2' \le \omega^2 + 3\omega + 2 - \sigma. \tag{40}$$

However, since

$$(\varepsilon_1 - 2)(\varepsilon_2' - 2) \ge 0,$$

it follows that

$$\varepsilon_1\varepsilon_2' \ge 2(\varepsilon_1 + \varepsilon_2') - 4. \tag{41}$$

Therefore, combining this with (34), we obtain

$$2\varepsilon_1\varepsilon_2' \ge 4(\omega + 2) - 8 = 4\omega.$$

Hence,

$$\omega^2 + 3\omega + 2 - \sigma \ge 4\omega,$$

and so

$$\omega^2 - \omega + 2 \ge \sigma,$$

as required.

15.8.1. *example.* If the type is $[2\nu_1 * 2\nu_1; \nu_1^7, \nu_1 - 1, \varepsilon, 2]$ where $\nu_1 > \varepsilon > 2$, then by

- $Y' = \nu_1 - 1 + \varepsilon + 2 = \nu_1 + \omega_1$,
- $X' = (\nu_1 - 1)^2 + \varepsilon^2 + 4 = \nu_1^2 + \omega_1 - 2\overline{g}$

we get

$$\omega_1 = \varepsilon + 1,$$

and so

$$\omega = \varepsilon + 1 + \overline{g}.$$

Moreover,

$$-2\nu_1 + 1 + \varepsilon^2 + 4 = \omega_1 - 2\overline{g}.$$

Hence,

$$\sigma = 2\nu_1 = \varepsilon^2 + 5 - (\omega_1 - 2\overline{g});$$

thus,

$$\sigma = (\omega_1 - 1)^2 + 5 - (\omega_1 - 2\overline{g});$$

where $g = \nu_1 - 1 - \frac{\varepsilon(\varepsilon-1)}{2}$.

Suppose that $g = 0$,i.e., $\nu_1 = 1 + \frac{\varepsilon(\varepsilon-1)}{2}$. Then $\omega = \varepsilon$ and

$$\sigma = \omega^2 - \omega + 2 = \alpha^2 + \alpha + 2.$$

If g =1 then

$$\sigma = \omega^2 - 3\omega + 6 = \alpha^2 - 3\alpha + 6.$$

16. Appendix

16.1. pairs with $\omega = 1, 2$.

Under the assumption $\sigma \geq 7$, we show the list of types of pairs with $\omega = 1, 2, 3, 4, 5, 6$. However, associated types are omitted, for simplicity.

Table 51. $\omega = 1, 2$

ω	σ	type	genus
1	7	$[7 * 9, 1; 1]$	27
2	7	$[7 * 9, 1; 2]$	26
2	8	$[8 * 8; 4^7]$	7
2	8	$[8 * 8; 4^7, 3]$	4
2	8	$[8 * 8; 4^7, 3^2]$	1
2	10	$[10 * 11; 5^9]$	0
2	12	$[12 * 12; 6^7, 5, 4]$	0

16.2. pairs with $\omega = 3$.

Table 52. $\omega = 3$

ω	σ	type	genus
3	7	$[7 * 9, 1; 2^2]$	25
3	8	$[8 * 9; 4^9]$	2
3	8	$[8 * 8; 4^7, 2]$	6
3	8	$[8 * 8; 4^7, 3, 2]$	3
3	8	$[8 * 8; 4^7, 3^2, 2]$	0
3	10	$[10 * 10; 5^7, 4, 3]$	2
3	10	$[10 * 10; 5^7, 4]$	5
3	12	$[12 * 12; 6^6, 5^3]$	1
3	14	$[14 * 14; 7^7, 6, 4]$	1
3	15	$[15 * 22, 1; 7^9]$	0
3	16	$[16 * 16; 8^6, 7^2, 6]$	0
3	20	$[20 * 20; 10^7, 9, 5]$	0

16.3. **pairs with $\omega = 4$.**

TABLE 53. $\omega = 4$

ω	σ	type	genus
4	7	$[7*9,1;2^3]$	24
4	8	$[8*9;4^9,2]$	1
4	8	$[8*8;4^6]$	13
4	8	$[8*8;4^6,3]$	10
4	8	$[8*8;4^6,3^2]$	7
4	8	$[8*8;4^6,3^3]$	4
4	8	$[8*8;4^6,3^4]$	1
4	8	$[8*8;4^7,2^2]$	5
4	8	$[8*8;4^7,3,2^2]$	2
4	9	$[9*13,1;4^{10}]$	0
4	10	$[10*10;5^7,4,2]$	4
4	10	$[10*10;5^7,4,3,2]$	1
4	10	$[10*10;5^6,4^3]$	3
4	10	$[10*10;5^6,4^3,3]$	0
4	12	$[12*13;6^8,5]$	2
4	12	$[12*12;6^6,5^3,2]$	0
4	12	$[12*12;6^7,5]$	6
4	12	$[12*12;6^7,5,3]$	3
4	12	$[12*12;6^7,5,3^2]$	0
4	14	$[14*14;7^7,5^2]$	2
4	14	$[14*14;7^7,6,4,2]$	0
4	16	$[16*16;8^7,7,4]$	2
4	16	$[16*16;8^5,7^4]$	1
4	16	$[16*17;8^8,6]$	1
4	18	$[18*18;9^7,7,6]$	1
4	19	$[19*19;9^9]$	0
4	19	$[19*38,2;9^9]$	0
4	20	$[20*20;10^5,9^3,8]$	0
4	22	$[22*22;11^7,10,5]$	1
4	22	$[22*22;11^6,10^2,7]$	0
4	22	$[22*22;11^7,8^2]$	0
4	30	$[30*30;15^7,14,6]$	0

TABLE 54. $\omega = 5$

ω	σ	type	genus
5	7	$[7 * 10, 1; 3^{11}]$	0
5	7	$[7 * 10, 1; 3^{10}]$	3
5	7	$[7 * 10, 1; 3^{9}]$	6
5	7	$[7 * 10, 1; 3^{8}]$	9
5	7	$[7 * 10, 1; 3^{7}]$	12
5	7	$[7 * 10, 1; 3^{6}]$	15
5	7	$[7 * 10, 1; 3^{5}]$	18
5	7	$[7 * 10, 1; 3^{4}]$	21
5	7	$[7 * 10, 1; 3^{3}]$	24
5	7	$[7 * 10, 1; 3^{2}]$	27
5	7	$[7 * 10, 1; 3]$	30
5	7	$[7 * 10, 1; 1]$	33
5	7	$[7 * 9, 1; 2^{4}]$	23
5	8	$[8 * 8; 4^{6}, 3^{4}, 2]$	0
5	8	$[8 * 9; 4^{9}, 2^{2}]$	0
5	8	$[8 * 8; 4^{7}, 3, 2^{3}]$	1
5	8	$[8 * 9; 4^{8}, 3^{2}]$	2
5	8	$[8 * 8; 4^{6}, 3^{3}, 2]$	3
5	8	$[8 * 8; 4^{7}, 2^{3}]$	4
5	8	$[8 * 9; 4^{8}, 3]$	5
5	8	$[8 * 8; 4^{6}, 3^{2}, 2]$	6
5	8	$[8 * 9; 4^{8}]$	8

16.4. **pairs with $\omega = 5$ (1).**

16.5. pairs with $\omega = 5$ (2).

TABLE 55. $\omega = 5$ (2)

ω	σ	type	genus
5	10	$[10 * 10; 5^7, 4, 3, 2^2]$	0
5	10	$[10 * 10; 5^5, 4^5]$	1
5	10	$[10 * 11; 5^8, 4, 3]$	1
5	10	$[10 * 10; 5^6, 4^3, 2]$	2
5	10	$[10 * 10; 5^7, 3^3]$	2
5	10	$[10 * 10; 5^7, 4, 2^2]$	3
5	10	$[10 * 11; 5^8, 4]$	4
5	10	$[10 * 10; 5^7, 3^2]$	5
5	10	$[10 * 10; 5^7, 3]$	8
5	10	$[10 * 10; 5^7]$	11
5	11	$[11 * 11; 5^{10}]$	0
5	11	$[11 * 22, 2; 5^{10}]$	0
5	12	$[12 * 12; 6^5, 5^4, 4]$	0
5	12	$[12 * 13; 6^8, 4^2]$	0
5	12	$[12 * 12; 6^7, 4^2, 3]$	1
5	12	$[12 * 13; 6^8, 5, 2]$	1
5	12	$[12 * 12; 6^7, 5, 3, 2]$	2
5	12	$[12 * 12; 6^7, 4^2]$	4
5	12	$[12 * 12; 6^7, 5, 2]$	5
5	13	$[13 * 19, 1; 6^9, 3]$	0
5	13	$[13 * 19, 1; 6^9]$	3
5	14	$[14 * 14; 7^6, 6^2, 5, 3]$	0
5	14	$[14 * 14; 7^7, 5, 4^2]$	0
5	14	$[14 * 14; 7^7, 5^2, 2]$	1
5	14	$[14 * 14; 7^7, 6, 3^2]$	1
5	14	$[14 * 14; 7^6, 6^2, 5]$	3
5	14	$[14 * 14; 7^7, 6, 3]$	4
5	14	$[14 * 14; 7^7, 6]$	7

16.6. pairs with $\omega = 5$ (3).

TABLE 56. $\omega = 5$ (2)

ω	σ	type	genus
5	16	$[16 * 16; 8^5, 7^4, 2]$	0
5	16	$[16 * 17; 8^8, 6, 2]$	0
5	16	$[16 * 16; 8^7, 7, 4, 2]$	1
5	16	$[16 * 17; 8^7, 7^2]$	2
5	18	$[18 * 18; 9^7, 7, 6, 2]$	0
5	18	$[18 * 18; 9^7, 8, 4, 3]$	0
5	18	$[18 * 18; 9^6, 8^2, 6]$	2
5	18	$[18 * 18; 9^7, 8, 4]$	3
5	20	$[20 * 20; 10^4, 9^5]$	1
5	20	$[20 * 21; 10^7, 9, 8]$	1
5	22	$[22 * 22; 11^7, 10, 5, 2]$	0
5	22	$[22 * 23; 11^8, 7]$	1
5	23	$[23 * 35, 1; 11^9]$	0
5	24	$[24 * 24; 12^4, 11^4, 10]$	0
5	24	$[24 * 25; 12^7, 10^2]$	0
5	24	$[24 * 24; 12^7, 10, 7]$	1
5	24	$[24 * 24; 12^7, 11, 5]$	2
5	25	$[25 * 37, 1; 12^8, 9]$	0
5	26	$[26 * 26; 13^6, 12, 11, 9]$	0
5	28	$[28 * 29; 14^8, 8]$	0
5	30	$[30 * 30; 15^7, 13, 8]$	0
5	32	$[32 * 32; 16^7, 15, 6]$	1
5	42	$[42 * 42; 21^7, 20, 7]$	0

16.7. pairs with $\omega = 6$ (1).

Table 57. $\omega = 6$

ω	σ	type	genus
6	7	$[7 * 10, 1; 3^{10}, 2]$	2
6	7	$[7 * 10, 1; 3^{9}, 2]$	5
6	7	$[7 * 10, 1; 3^{8}, 2]$	8
6	7	$[7 * 9, 1; 2^{5}]$	22
6	8	$[8 * 8; 4^{7}, 3, 2^{4}]$	0
6	8	$[8 * 10; 4^{10}, 3]$	0
6	8	$[8 * 8; 4^{5}, 3^{6}]$	1
6	8	$[8 * 9; 4^{8}, 3^{2}, 2]$	1
6	8	$[8 * 8; 4^{6}, 3^{3}, 2^{2}]$	2
6	8	$[8 * 8; 4^{7}, 2^{4}]$	3
6	8	$[8 * 10; 4^{10}]$	3
6	8	$[8 * 8; 4^{5}, 3^{5}]$	4
6	8	$[8 * 9; 4^{8}, 3, 2]$	4
6	8	$[8 * 8; 4^{6}, 3^{2}, 2^{2}]$	5
6	8	$[8 * 8; 4^{5}, 3^{4}]$	7
6	8	$[8 * 9; 4^{8}, 2]$	7
6	8	$[8 * 8; 4^{6}, 3, 2^{2}]$	8
6	8	$[8 * 10, 1; 1]$	35
6	9	$[9 * 13, 1; 4^{9}, 3^{2}]$	0
6	9	$[9 * 13, 1; 4^{9}, 3]$	3
6	9	$[9 * 13, 1; 4^{9}]$	6

TABLE 58. $\omega = 6$

ω	σ	type	genus
6	10	$[10 * 10; 5^5, 4^5, 2]$	0
6	10	$[10 * 10; 5^6, 4^2, 3^3]$	0
6	10	$[10 * 11; 5^8, 4, 3, 2]$	0
6	10	$[10 * 10; 5^6, 4^3, 2^2]$	1
6	10	$[10 * 10; 5^7, 3^3, 2]$	1
6	10	$[10 * 10; 5^7, 4, 2^3]$	2
6	10	$[10 * 11; 5^7, 4^3]$	2
6	10	$[10 * 10; 5^6, 4^2, 3^2]$	3
6	10	$[10 * 11; 5^8, 4, 2]$	3
6	10	$[10 * 10; 5^7, 3^2, 2]$	4
6	10	$[10 * 10; 5^6, 4^2, 3]$	6
6	10	$[10 * 10; 5^7, 3, 2]$	7
6	11	$[11 * 16, 1; 5^9, 3]$	2
6	11	$[11 * 16, 1; 5^9]$	5
6	12	$[12 * 12; 6^7, 4^2, 3, 2]$	0
6	12	$[12 * 13; 6^8, 5, 2^2]$	0
6	12	$[12 * 12; 6^7, 5, 3, 2^2]$	1
6	12	$[12 * 12; 6^4, 5^6]$	1
6	12	$[12 * 13; 6^7, 5^2, 4]$	1
6	12	$[12 * 12; 6^6, 5^2, 4, 3]$	2
6	12	$[12 * 12; 6^7, 4^2, 2]$	3
6	12	$[12 * 12; 6^7, 5, 2^2]$	4
6	12	$[12 * 12; 6^6, 5^2, 4]$	5
6	13	$[13 * 20, 1; 6^{10}]$	0
6	13	$[13 * 19, 1; 6^9, 2]$	2

16.8. **pairs with $\omega = 6$ (2).**

TABLE 59. $\omega = 6$

ω	σ	type	genus
6	14	$[14 * 14; 7^7, 5^2, 2^2]$	0
6	14	$[14 * 14; 7^7, 6, 3^2, 2]$	0
6	14	$[14 * 14; 7^4, 6^5, 5]$	0
6	14	$[14 * 15; 7^7, 6, 5^2]$	0
6	14	$[14 * 16; 7^9, 4]$	0
6	14	$[14 * 14; 7^5, 6^4, 3]$	1
6	14	$[14 * 14; 7^6, 6^2, 4^2]$	1
6	14	$[14 * 15; 7^8, 5, 3]$	1
6	14	$[14 * 14; 7^6, 6^2, 5, 2]$	2
6	14	$[14 * 14; 7^7, 6, 3, 2]$	3
6	14	$[14 * 14; 7^5, 6^4]$	4
6	14	$[14 * 15; 7^8, 5]$	4
6	14	$[14 * 14; 7^7, 6, 2]$	6
6	15	$[15 * 22, 1; 7^8, 6, 4]$	0
6	16	$[16 * 16; 8^7, 7, 4, 2^2]$	0
6	16	$[16 * 16; 8^6, 7, 6^2, 4]$	0
6	16	$[16 * 17; 8^8, 5, 4]$	0
6	16	$[16 * 18; 8^9, 3]$	0
6	16	$[16 * 16; 8^7, 6, 5, 3]$	1
6	16	$[16 * 17; 8^7, 7^2, 2]$	1
6	16	$[16 * 16; 8^7, 7, 3^2]$	2
6	16	$[16 * 18; 8^9]$	3
6	16	$[16 * 16; 8^7, 6, 5]$	4
6	16	$[16 * 16; 8^7, 7, 3]$	5

16.9. **pairs with $\omega = 6$ (3).**

Table 60. $\omega = 6$ (2)

ω	σ	type	genus
6	17	$[17*25,1;8^8,7,3]$	0
6	17	$[17*25,1;8^8,7]$	3
6	18	$[18*18;9^6,8,7^2,3]$	0
6	18	$[18*18;9^7,7,5,4]$	0
6	18	$[18*19;9^8,6,3]$	0
6	18	$[18*18;9^6,8^2,6,2]$	1
6	18	$[18*18;9^7,8,4,2]$	2
6	18	$[18*18;9^6,8,7^2]$	3
6	18	$[18*19;9^8,6]$	3
6	20	$[20*20;10^4,9^5,2]$	0
6	20	$[20*20;10^7,8,6,3]$	0
6	20	$[20*21;10^7,9,8,2]$	0
6	20	$[20*20;10^7,9,4,3]$	1
6	20	$[20*21;10^6,9^3]$	2
6	20	$[20*20;10^7,8,6]$	3
6	20	$[20*20;10^7,9,4]$	4
6	21	$[21*31,1;10^8,8]$	2
6	22	$[22*23;11^8,7,2]$	0
6	22	$[22*22;11^6,10,9,8]$	2
6	24	$[24*24;12^7,10,7,2]$	0
6	24	$[24*24;12^7,11,4^2]$	0
6	24	$[24*24;12^7,11,5,2]$	1
6	24	$[24*24;12^3,11^6]$	1
6	24	$[24*25;12^6,11^2,10]$	1
6	24	$[24*24;12^6,11^2,7]$	2
6	26	$[26*26;13^7,12,5,3]$	0
6	26	$[26*26;13^5,12^3,9]$	1
6	26	$[26*26;13^6,12,10^2]$	1
6	26	$[26*26;13^7,12,5]$	3
6	27	$[27*28;13^9]$	0
6	27	$[27*55,2;13^9]$	0
6	28	$[28*41,1;13^9]$	0
6	28	$[28*28;14^3,13^5,12]$	0
6	28	$[28*29;14^6,13,12^2]$	0
6	28	$[28*30;14^8,11]$	0
6	28	$[28*28;14^7,11,9]$	1
6	29	$[29*43,1;14^7,13,11]$	0
6	30	$[30*30;15^6,13^2,11]$	0
6	30	$[30*30;15^6,14^2,8]$	1
6	31	$[31*46,1;15^8,10]$	0
6	32	$[32*32;16^7,15,6,2]$	0
6	32	$[32*32;16^6,15,14,10]$	0
6	32	$[32*32;16^7,12,11]$	0
6	34	$[34*34;17^7,16,6]$	2
6	36	$[36*37;18^8,9]$	0
6	38	$[38*38;19^7,17,9]$	0
6	44	$[44*44;22^7,21,7]$	1
6	56	$[56*56;28^7,27,8]$	0

17. PAIRS WITH SMALL α

TABLE 61. $\alpha = 1, 2, 3$

α	σ	type	genus
1	10	$[10 * 11; 5^9]$	0
1	12	$[12 * 12; 6^7, 5, 4]$	0
2	8	$[8 * 8; 4^7, 3^2, 2]$	0
2	8	$[8 * 8; 4^7, 3^2]$	1
2	15	$[15 * 22, 1; 7^9]$	0
2	16	$[16 * 16; 8^6, 7^2, 6]$	0
2	20	$[20 * 20; 10^7, 9, 5]$	0
3	9	$[9 * 13, 1; 4^{10}]$	0
3	10	$[10 * 10; 5^6, 4^3, 3]$	0
3	12	$[12 * 12; 6^6, 5^3, 2]$	0
3	12	$[12 * 12; 6^7, 5, 3^2]$	0
3	12	$[12 * 12; 6^6, 5^3]$	1
3	14	$[14 * 14; 7^7, 6, 4, 2]$	0
3	14	$[14 * 14; 7^7, 6, 4]$	1
3	19	$[19 * 19; 9^9]$	0
3	19	$[19 * 38, 2; 9^9]$	0
3	20	$[20 * 20; 10^5, 9^3, 8]$	0
3	22	$[22 * 22; 11^6, 10^2, 7]$	0
3	22	$[22 * 22; 11^7, 8^2]$	0
3	30	$[30 * 30; 15^7, 14, 6]$	0

17.1. pairs with $\alpha = 1, 2, 3$.

Table 62. $\alpha = 4$

α	σ	type	genus
4	7	$[7*10,1;3^{11}]$	0
4	8	$[8*8;4^6,3^4,2]$	0
4	8	$[8*9;4^9,2^2]$	0
4	8	$[8*8;4^6,3^4]$	1
4	8	$[8*9;4^9,2]$	1
4	8	$[8*9;4^9]$	2
4	10	$[10*10;5^7,4,3,2^2]$	0
4	10	$[10*10;5^7,4,3,2]$	1
4	10	$[10*10;5^7,4,3]$	2
4	11	$[11*11;5^{10}]$	0
4	11	$[11*22,2;5^{10}]$	0
4	12	$[12*12;6^5,5^4,4]$	0
4	12	$[12*13;6^8,4^2]$	0
4	13	$[13*19,1;6^9,3]$	0
4	14	$[14*14;7^6,6^2,5,3]$	0
4	14	$[14*14;7^7,5,4^2]$	0
4	16	$[16*16;8^5,7^4,2]$	0
4	16	$[16*17;8^8,6,2]$	0
4	16	$[16*16;8^5,7^4]$	1
4	16	$[16*17;8^8,6]$	1
4	18	$[18*18;9^7,7,6,2]$	0
4	18	$[18*18;9^7,8,4,3]$	0
4	18	$[18*18;9^7,7,6]$	1
4	22	$[22*22;11^7,10,5,2]$	0
4	22	$[22*22;11^7,10,5]$	1
4	23	$[23*35,1;11^9]$	0
4	24	$[24*24;12^4,11^4,10]$	0
4	24	$[24*25;12^7,10^2]$	0
4	25	$[25*37,1;12^8,9]$	0
4	26	$[26*26;13^6,12,11,9]$	0
4	28	$[28*29;14^8,8]$	0
4	30	$[30*30;15^7,13,8]$	0
4	42	$[42*42;21^7,20,7]$	0

17.2. **pairs with $\alpha = 4$.**

TABLE 63. $\alpha = 5$

α	σ	type	genus
5	8	$[8 * 8; 4^7, 3, 2^4]$	0
5	8	$[8 * 10; 4^{10}, 3]$	0
5	8	$[8 * 8; 4^7, 3, 2^3]$	1
5	8	$[8 * 8; 4^7, 3, 2^2]$	2
5	8	$[8 * 8; 4^7, 3, 2]$	3
5	8	$[8 * 8; 4^7, 3]$	4
5	9	$[9 * 13, 1; 4^9, 3^2]$	0
5	10	$[10 * 10; 5^5, 4^5, 2]$	0
5	10	$[10 * 10; 5^6, 4^2, 3^3]$	0
5	10	$[10 * 11; 5^8, 4, 3, 2]$	0
5	10	$[10 * 10; 5^5, 4^5]$	1
5	10	$[10 * 11; 5^8, 4, 3]$	1
5	12	$[12 * 12; 6^7, 4^2, 3, 2]$	0
5	12	$[12 * 13; 6^8, 5, 2^2]$	0
5	12	$[12 * 12; 6^7, 4^2, 3]$	1
5	12	$[12 * 13; 6^8, 5, 2]$	1
5	12	$[12 * 13; 6^8, 5]$	2
5	13	$[13 * 20, 1; 6^{10}]$	0
5	14	$[14 * 14; 7^7, 5^2, 2^2]$	0
5	14	$[14 * 14; 7^7, 6, 3^2, 2]$	0
5	14	$[14 * 14; 7^4, 6^5, 5]$	0
5	14	$[14 * 15; 7^7, 6, 5^2]$	0
5	14	$[14 * 16; 7^9, 4]$	0
5	14	$[14 * 14; 7^7, 5^2, 2]$	1
5	14	$[14 * 14; 7^7, 6, 3^2]$	1
5	14	$[14 * 14; 7^7, 5^2]$	2
5	15	$[15 * 22, 1; 7^8, 6, 4]$	0

17.3. **pairs with $\alpha = 5$.**

TABLE 64. $\alpha = 5, continued$

α	σ	type	genus
5	16	$[16 * 16; 8^7, 7, 4, 2^2]$	0
5	16	$[16 * 16; 8^6, 7, 6^2, 4]$	0
5	16	$[16 * 17; 8^8, 5, 4]$	0
5	16	$[16 * 18; 8^9, 3]$	0
5	16	$[16 * 16; 8^7, 7, 4, 2]$	1
5	16	$[16 * 16; 8^7, 7, 4]$	2
5	17	$[17 * 25, 1; 8^8, 7, 3]$	0
5	18	$[18 * 18; 9^6, 8, 7^2, 3]$	0
5	18	$[18 * 18; 9^7, 7, 5, 4]$	0
5	18	$[18 * 19; 9^8, 6, 3]$	0
5	20	$[20 * 20; 10^4, 9^5, 2]$	0
5	20	$[20 * 20; 10^7, 8, 6, 3]$	0
5	20	$[20 * 21; 10^7, 9, 8, 2]$	0
5	20	$[20 * 20; 10^4, 9^5]$	1
5	20	$[20 * 21; 10^7, 9, 8]$	1
5	22	$[22 * 23; 11^8, 7, 2]$	0
5	22	$[22 * 23; 11^8, 7]$	1
5	24	$[24 * 24; 12^7, 10, 7, 2]$	0
5	24	$[24 * 24; 12^7, 11, 4^2]$	0
5	24	$[24 * 24; 12^7, 10, 7]$	1
5	26	$[26 * 26; 13^7, 12, 5, 3]$	0
5	27	$[27 * 28; 13^9]$	0
5	27	$[27 * 55, 2; 13^9]$	0
5	28	$[28 * 41, 1; 13^9]$	0
5	28	$[28 * 28; 14^3, 13^5, 12]$	0
5	28	$[28 * 29; 14^6, 13, 12^2]$	0
5	28	$[28 * 30; 14^8, 11]$	0
5	29	$[29 * 43, 1; 14^7, 13, 11]$	0
5	30	$[30 * 30; 15^6, 13^2, 11]$	0
5	31	$[31 * 46, 1; 15^8, 10]$	0
5	32	$[32 * 32; 16^7, 15, 6, 2]$	0
5	32	$[32 * 32; 16^6, 15, 14, 10]$	0
5	32	$[32 * 32; 16^7, 12, 11]$	0
5	32	$[32 * 32; 16^7, 15, 6]$	1
5	36	$[36 * 37; 18^8, 9]$	0
5	38	$[38 * 38; 19^7, 17, 9]$	0
5	56	$[56 * 56; 28^7, 27, 8]$	0

17.4. **pairs with** $\alpha = 5, continued$**.**

TABLE 65. $\alpha = 6, (1)$

α	σ	type	genus
6	7	$[7 * 14, 2; 3^{12}]$	0
6	7	$[7 * 7; 3^{12}]$	0
6	8	$[8 * 8; 4^5, 3^6, 2]$	0
6	8	$[8 * 9; 4^8, 3^2, 2^2]$	0
6	8	$[8 * 8; 4^5, 3^6]$	1
6	8	$[8 * 9; 4^8, 3^2, 2]$	1
6	8	$[8 * 9; 4^8, 3^2]$	2
6	10	$[10 * 10; 5^6, 4^3, 2^3]$	0
6	10	$[10 * 10; 5^7, 3^3, 2^2]$	0
6	10	$[10 * 11; 5^6, 4^5]$	0
6	10	$[10 * 12; 5^9, 4, 3]$	0
6	10	$[10 * 10; 5^6, 4^3, 2^2]$	1
6	10	$[10 * 10; 5^7, 3^3, 2]$	1
6	10	$[10 * 10; 5^6, 4^3, 2]$	2
6	10	$[10 * 10; 5^7, 3^3]$	2
6	10	$[10 * 10; 5^6, 4^3]$	3
6	11	$[11 * 16, 1; 5^8, 4^2, 3]$	0
6	12	$[12 * 12; 6^7, 5, 3, 2^3]$	0
6	12	$[12 * 12; 6^4, 5^6, 2]$	0
6	12	$[12 * 12; 6^5, 5^4, 3^2]$	0
6	12	$[12 * 12; 6^6, 5, 4^3, 3]$	0
6	12	$[12 * 13; 6^7, 5^2, 4, 2]$	0
6	12	$[12 * 13; 6^8, 4, 3^2]$	0
6	12	$[12 * 12; 6^7, 5, 3, 2^2]$	1
6	12	$[12 * 12; 6^4, 5^6]$	1
6	12	$[12 * 13; 6^7, 5^2, 4]$	1
6	12	$[12 * 12; 6^7, 5, 3, 2]$	2
6	12	$[12 * 12; 6^7, 5, 3]$	3

17.5. **pairs with $\alpha = 6, (1)$.**

TABLE 66. $\alpha = 6, (2)$

α	σ	type	genus
6	14	$[14 * 14; 7^5, 6^4, 3, 2]$	0
6	14	$[14 * 14; 7^6, 6^2, 4^2, 2]$	0
6	14	$[14 * 14; 7^7, 5, 4, 3^2]$	0
6	14	$[14 * 15; 7^8, 5, 3, 2]$	0
6	14	$[14 * 14; 7^5, 6^4, 3]$	1
6	14	$[14 * 14; 7^6, 6^2, 4^2]$	1
6	14	$[14 * 15; 7^8, 5, 3]$	1
6	15	$[15 * 16; 7^{10}]$	0
6	15	$[15 * 31, 2; 7^{10}]$	0
6	16	$[16 * 23, 1; 7^{10}]$	0
6	16	$[16 * 16; 8^7, 6, 5, 3, 2]$	0
6	16	$[16 * 17; 8^7, 7^2, 2^2]$	0
6	16	$[16 * 16; 8^3, 7^6, 6]$	0
6	16	$[16 * 17; 8^6, 7^2, 6^2]$	0
6	16	$[16 * 18; 8^8, 7, 5]$	0
6	16	$[16 * 16; 8^7, 6, 5, 3]$	1
6	16	$[16 * 17; 8^7, 7^2, 2]$	1
6	16	$[16 * 17; 8^7, 7^2]$	2
6	17	$[17 * 25, 1; 8^7, 7^2, 5]$	0
6	18	$[18 * 18; 9^6, 8^2, 6, 2^2]$	0
6	18	$[18 * 18; 9^7, 8, 3^3]$	0
6	18	$[18 * 18; 9^5, 8^3, 6, 5]$	0
6	18	$[18 * 18; 9^6, 7^3, 5]$	0
6	18	$[18 * 18; 9^6, 8, 6^3]$	0
6	18	$[18 * 19; 9^6, 8^3, 4]$	0
6	18	$[18 * 18; 9^6, 8^2, 6, 2]$	1
6	18	$[18 * 18; 9^6, 8^2, 6]$	2
6	19	$[19 * 28, 1; 9^8, 7, 4]$	0

17.6. **pairs with** $\alpha = 6, (2)$.

TABLE 67. $\alpha = 6, (3)$

α	σ	type	genus
6	20	$[20 * 20; 10^7, 9, 4, 3, 2]$	0
6	20	$[20 * 20; 10^6, 9, 8, 7, 4]$	0
6	20	$[20 * 20; 10^7, 7, 6, 5]$	0
6	20	$[20 * 21; 10^8, 5^2]$	0
6	20	$[20 * 22; 10^8, 9, 3]$	0
6	20	$[20 * 20; 10^7, 9, 4, 3]$	1
6	21	$[21 * 31, 1; 10^7, 9^2, 3]$	0
6	22	$[22 * 22; 11^5, 10^3, 8, 3]$	0
6	22	$[22 * 22; 11^6, 10^2, 6, 4]$	0
6	22	$[22 * 22; 11^6, 9^3, 3]$	0
6	22	$[22 * 22; 11^7, 9, 5^2]$	0
6	24	$[24 * 24; 12^7, 11, 5, 2^2]$	0
6	24	$[24 * 24; 12^3, 11^6, 2]$	0
6	24	$[24 * 24; 12^7, 9, 8, 3]$	0
6	24	$[24 * 25; 12^6, 11^2, 10, 2]$	0
6	24	$[24 * 25; 12^8, 7, 3]$	0
6	24	$[24 * 24; 12^7, 11, 5, 2]$	1
6	24	$[24 * 24; 12^3, 11^6]$	1
6	24	$[24 * 25; 12^6, 11^2, 10]$	1
6	24	$[24 * 24; 12^7, 11, 5]$	2
6	26	$[26 * 26; 13^5, 12^3, 9, 2]$	0
6	26	$[26 * 26; 13^6, 12, 10^2, 2]$	0
6	26	$[26 * 26; 13^7, 11, 7, 3]$	0
6	26	$[26 * 26; 13^5, 12^3, 9]$	1
6	26	$[26 * 26; 13^6, 12, 10^2]$	1
6	28	$[28 * 28; 14^7, 11, 9, 2]$	0
6	28	$[28 * 28; 14^7, 11, 9]$	1

Table 68. $\alpha = 6, (4)$

α	σ	type	genus
6	30	$[30 * 30; 15^6, 14^2, 8, 2]$	0
6	30	$[30 * 30; 15^6, 14^2, 8]$	1
6	31	$[31 * 48, 1; 15^9]$	0
6	32	$[32 * 32; 16^7, 15, 5, 4]$	0
6	32	$[32 * 32; 16^2, 15^6, 14]$	0
6	32	$[32 * 33; 16^5, 15^2, 14^2]$	0
6	32	$[32 * 34; 16^7, 15, 13]$	0
6	33	$[33 * 49, 1; 16^6, 15^2, 13]$	0
6	34	$[34 * 34; 17^5, 16^2, 14, 13]$	0
6	34	$[34 * 34; 17^6, 14^3]$	0
6	34	$[34 * 35; 17^6, 16^2, 12]$	0
6	36	$[36 * 36; 18^7, 17, 6, 3]$	0
6	36	$[36 * 36; 18^6, 17, 15, 12]$	0
6	38	$[38 * 38; 19^5, 18^3, 11]$	0
6	40	$[40 * 40; 20^7, 17, 11]$	0
6	44	$[44 * 44; 22^7, 21, 7, 2]$	0
6	44	$[44 * 44; 22^7, 21, 7]$	1
6	46	$[46 * 46; 23^6, 22^2, 10]$	0
6	72	$[72 * 72; 36^7, 35, 9]$	0

18. BIBLIOGRAPHY

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