KODAIRA DIMENSION

AND MIXEDPLURIGENERA

OF ALGEBRAIC VARIETIES

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1. ELLIPSES, HYPERBOLAS, HYPERBOLICS

Algebraic plane curves were treated Apollonius of Perga (died c.190 BC) in the study of conic sections which are ellipses , the parabolas, and hyperbolas.

Rene' Descartes applied his newly discovered Analytic geometry to the study of conics. This had the effect of reducing the geometrical problems of conics to problems in algebra. Features Conics are of three types: parabolas, ellipses, including circles, and hyperbolas.

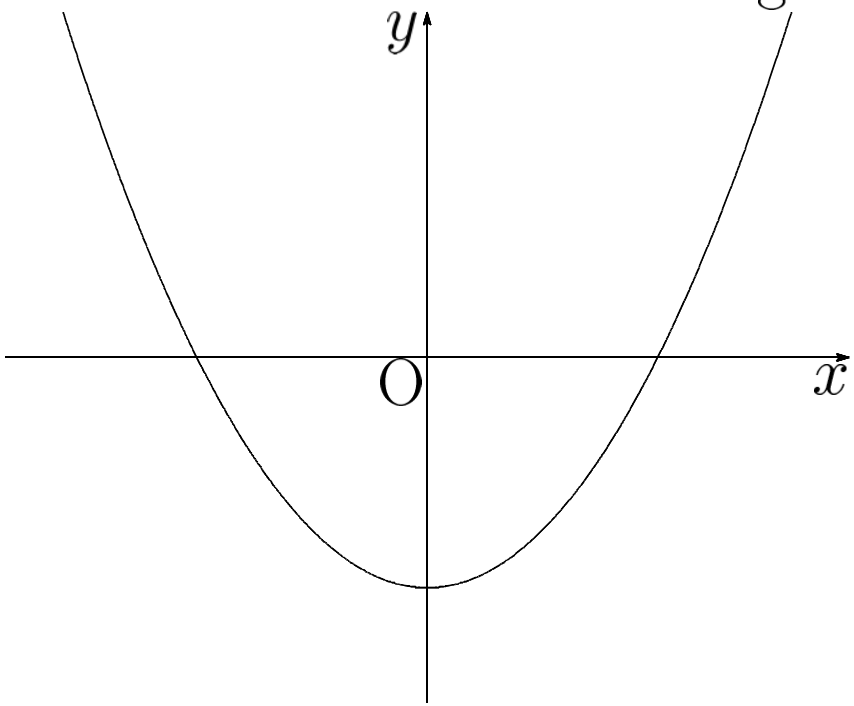
A conic is defined by $x^2 + y^2 = z^2$. Cutting out by a plane $ax+by+cz = d$, one gets the quadratic equation by eliminating the variable z.

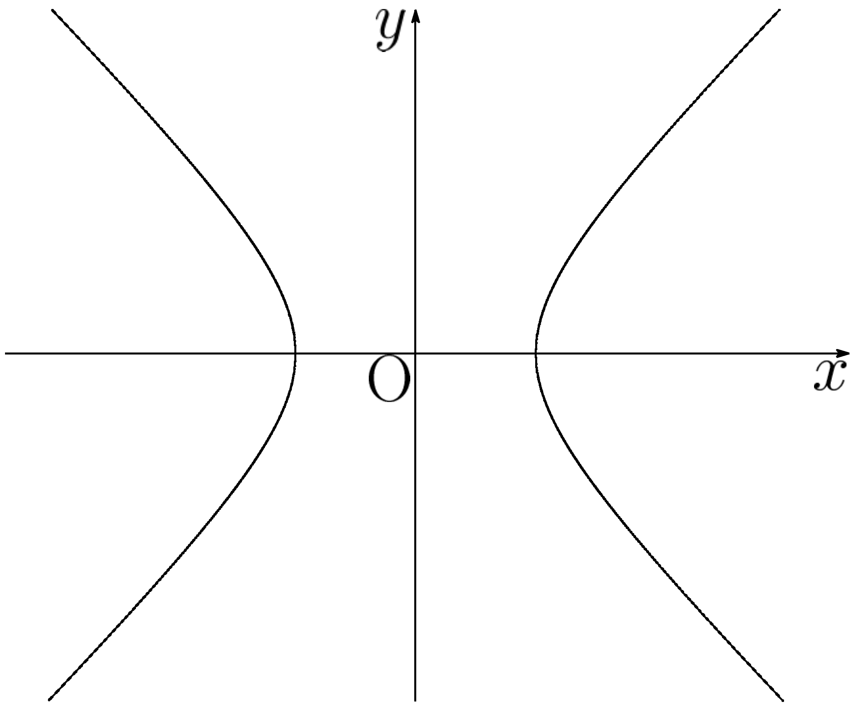
$$ax^2 + 2hxy + by^2 + 2ex + 2fy + c = 0$$

By applying Euclidean transformation

- $x' = ax + by + p$,
- $y' = cx + dy + q$

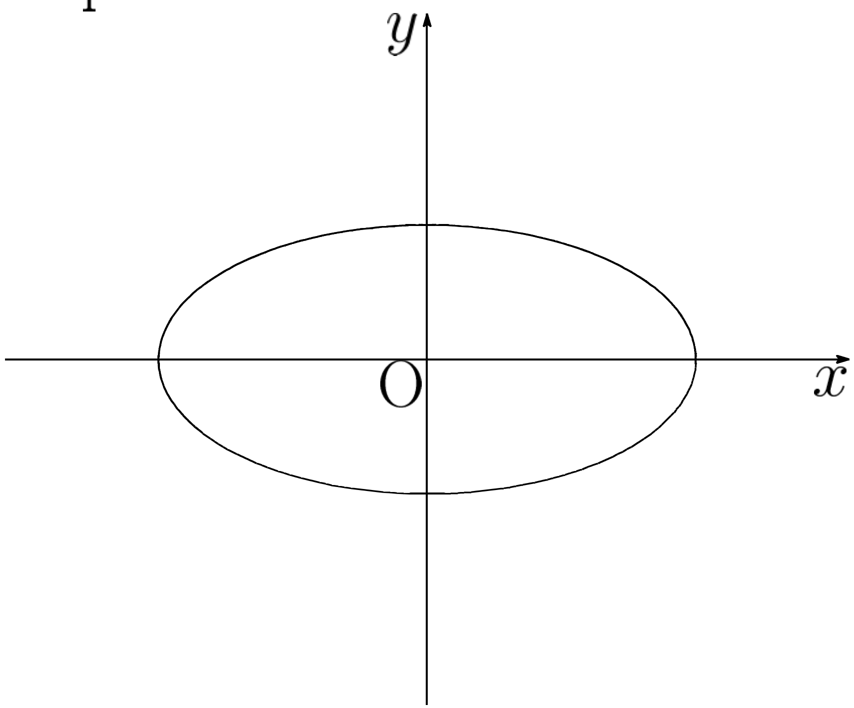
$A = \begin{pmatrix} a & b & c & d \end{pmatrix}$, A being a orthogonal matrix, conics are transformed into one of the followings.





hyperbola

ellipse



1.1. **imaginary circles.** By definition, one has a curve defined by $x^2 + y^2 = -1$, which looks like empty. But one cannot exclude such quadratic equations. Sometimes the equation defines imaginary circle. In order to treat these , one has to points whose coordinates consist of complex numbers. Then a figure defined by $x^2 + y^2 = 1$ is pairs of complex numbers turns out to be a real surface in the 4-dimensional space. By using complex numbers, one can apply algebraic treatment easily, neglecting visual difficulties.

2. PROJECTIVE CURVES

Conics have three different types. But from the view point of projective geometry, conics are just one thing.

Instead of usual coordinates (x, y), one introduce homogeneous coordinates x_0, x_1, x_2, by putting

$$x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}.$$

For example, a parabola $y = x^2$ is transformed into $x_2 x_0 = x_1^2$ by using homogeneous coordinates.

Applying a suitable projective linear transformation, one gets ${x_0}^2 = x_1^2 + x_2^2$. It is clear that any conics are transformed into ${x_0}^2 = x_1^2 + x_2^2$ by a projective linear transformation.

Linear transformations using homogeneous coordinates is called projective transformations. As usual, coefficients are complex numbers. Then one has complex projective transformations.

By using usual coordinates $x = x_1/x_0, y = x_2/x_0$ and $x' = x'_1/x'_0, y' = x'_2/x'_0$, projective complex numbers transformations are expressed as follows:

- $x' = \frac{ax+by+p}{a''x+b''y+p''}$,
- $y' = \frac{a'x+b'y+p'}{a''x+b''y+p''}$

where

$$A = \begin{pmatrix} a & b & p \\ a' & b' & p' \\ a'' & b'' & p'' \end{pmatrix},$$

A being complex regular matrix.

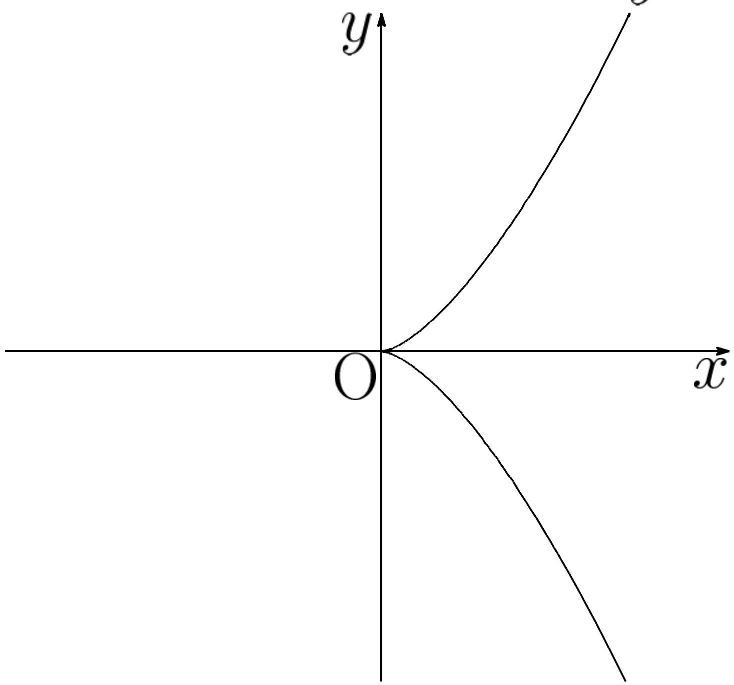
It is clear that any conics are transformed into ${x_0}^2 = x_1^2 + x_2^2$ by a projective linear transformation.

3. CUBIC CURVES

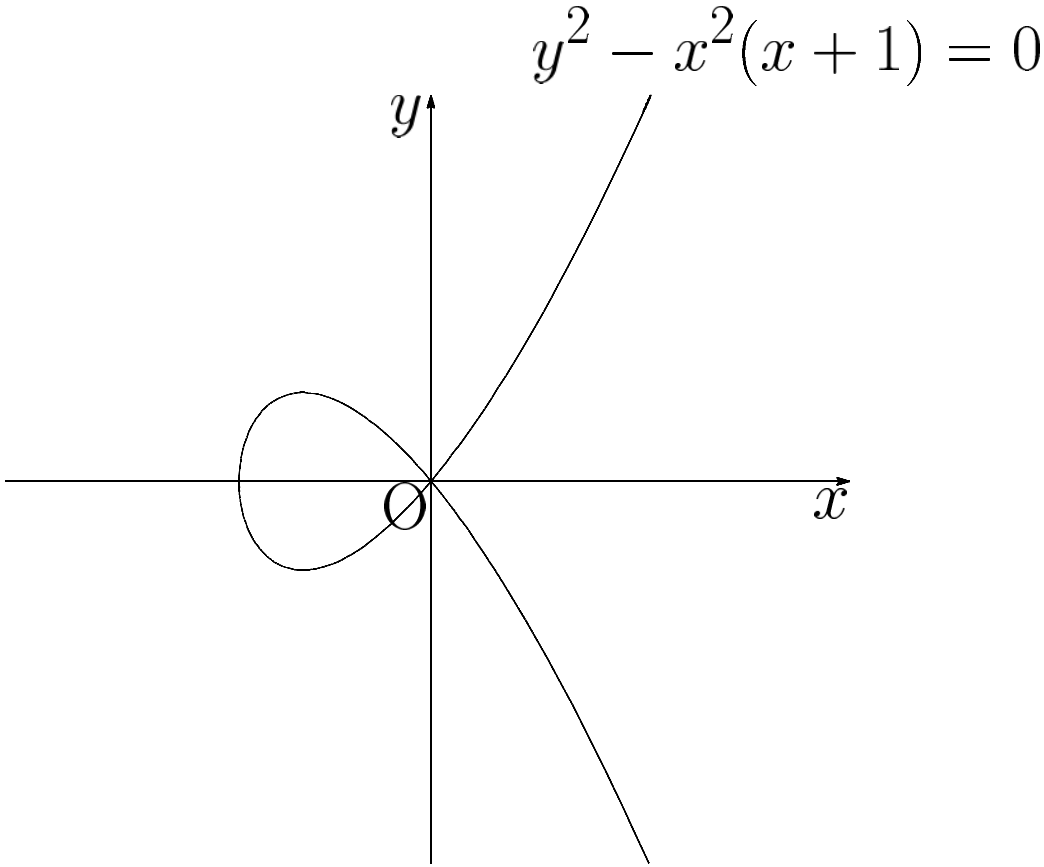
Cubic curves are transformed into the following three types of curves by complex projective transformations.

cuspidal cubic

$$y^2 - x^3 = 0$$

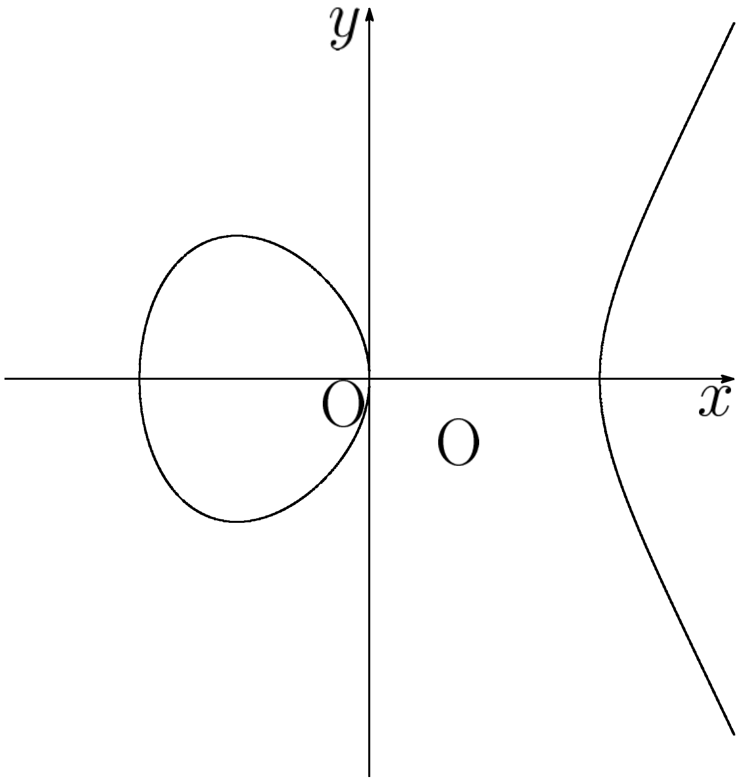


nodal cubic



non-singular elliptic curve

$$y^2 - x(x-1)(x+\lambda) = 0, \lambda \neq 0, -1$$



4. ALGEBRAIC PLANE CURVES

Any irreducible polynomials in two variables $f(x, y)$ define algebraic plane curves. The degree of the polynomial is the degree of the curve, which turns out to be the number of common points of the curve and a straight line, namely, a curve of degree 1.

FIGURE 1

5. RATIONAL CURVES

If the defining equation $f(x, y)$ is parametrized by rational functions is called a rational curve.

Rational curves are simple ones among algebraic curves.

In addition to degree, algebraic curves have another invariant called genus, which was introduced by Riemann.

algebraic curves are rational if and only the genus vanishes.

The genus is defined to be the maximal number of linearly independent regular 1-forms of the curve.

In order to define regular differential 1-forms, one has to a nonsingular curve, which is derived by resolving the singularity of the curve.

6. BIRATIONAL TRANSFORMATIONS

The following is a very simple example of birational transformations.

The system

- $x' = x$,
- $y' = y + x^2$

is rewritten as

- $x = x'$,
- $y = y' - x'^2$.

The parabola $y = -x^2$ is transformed into a line by the birational transformation.

In general, rational expressions $P(x, y), Q(x, y)$ define the rational transformation.

- $x' = P(x, y)$,
- $y' = Q(x, y)$

Moreover, if the inverse transformation exists, it is the birational transformation, called Cremona transformation, 6.1. **curves of degree 4.** Curves C of degree 4 are classified into the following four types according to Kodaira dimension $\kappa[D]$ and genus g,D being the nonsingular model of D.

- $g = 3$. Then C is non-singular and $\kappa = 2$. plane curves of general type.
- $g = 2$. Then C has a double point and $\kappa = 1$. By Cremona transformation, C is transformed into a hyperelliptic plane curve.
- $g = 1$. Then C has 2 double points and $\kappa = 0$. By Cremona transformation,C is transformed into a nonsingular cubic; an elliptic plane curve.
- $g = 0$. Then $\kappa = -\infty$. By Cremona transformation, C is transformed into a line.

Curves with degree greater than five are much more complicated, which will be explained later.

However, Using κ , they are classified into the following four types according to the value of κ, which take on of $2, 1, 0, -\infty$.

By the way, cubic curves have $\kappa = 0, -\infty$.

Here, we recall Kodaira dimension.

7. Q.T

The fundamental quadratic transformation T between the projective plane $\mathbf{P}^2$ is defined by

$$Y_0 = X_1 X_2, \quad Y_1 = X_0 X_2, \quad Y_2 = X_1 X_0.$$

By Q.T, a plane curve C of degree d is transformed into a plane curve C'.

FIGURE 2

Ex. If C is a quartic with three double points P_0, P_1, P_2, then C' turns out to be a conic.

7.1. **Noether's formula.** Suppose that C is a curve of degree d with singular points with multiplicites $\nu_0, \nu_1, \nu_2, \cdots, \nu_r$ among which $\nu_0 \geq \nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$.

If C has singular points with multiplicities ν_0, ν_1, ν_2 at P_0, P_1, P_2,then C' has multiplicities ${\nu_0}', \nu_1', {\nu_2}'$ at Q_0, Q_1, Q_2 where

$$\nu_0' = d - \nu_1 - \nu_2, \quad \nu_1' = d - \nu_0 - \nu_2, \quad \nu_2' = d - \nu_1 - \nu_2$$

.

Here, $d' = 2d - \nu_0 - \nu_1 - \nu_2$. (Noether's formula)

In particular, if $d' < d$,then C' looks much simpler than C.
$d' < d$ if and only $d < \nu_0 + \nu_1 + \nu_2$.
$d < \nu_0 + \nu_1 + \nu_2$ is called Noether's inequality.

However, the curve defined by $x = \cos 6\theta, y = \cos 5\theta$ has degree 6 and 10 double points.

Since $\nu_0 = \nu_1 = \nu_2 = 2$, it follows that $d' = 2d - \nu_0 - \nu_1 - \nu_2 = 12 - 6 = 6$.

Thus C' does not look simpler than C.

For a singular rational curve C on $\mathbf{P}^2$, after a finite number of blowing ups we get a nonsingular rational surface S and a nonsingular curve D which is the proper transform of C.

Say C has degree d and singular points with multiplicites $\nu_0, \nu_1, \nu_2, \cdots, \nu_r$ with $\nu_0 \geq \nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$.

Then

$$D^2 = d^2 - \nu_0^2 - \nu_1^2 - \cdots - \nu_r^2$$

In this case , we say that C has the numerical type $[d; \nu_0, \nu_1, \nu_2, \cdots, \nu_r]$.

Fact 3. If C satisfies that $D^2 \geq -3$, C is tranformed into a line on $\mathbf{P}^2$ by a Cremona transformation.

NB:

- Curves with type $[5; 2^6]$ satisfy $\kappa[D] = \infty, D^2 = 25 - 4 \cdot 6 = 1.$
- Curves with type $[6; 2^{10}]$ satisfy $\kappa[D] = 0, D^2 = 36 - 4 \cdot 10 = -4.$
- Curves with type $[7; 2^{15}]$ satisfy $\kappa[D] = 2, D^2 = 49 - 4 \cdot 15 = -11.$

Figure 3

7.2. **Nagata.** In 1960, M.Nagata(who passed away in 2008) showed that:

Let D be an elliptic curve on a nonsingular rational surface S

.

Then $D^2 \leq 9$.
If $D^2 = 9$ then (S, D) is transformed into $(C_3, \mathbf{P^2})$.
Here C_d means a nonsingular plane curve of degree d.

7.3. **Hartshorne.** In 1970,R.Hartshorne showed that if $g > 1$ then $D^2 \leq 4g + 4$.

Moreover, if hyperelliptic curves defined by $y^2 = \Pi_{j=1}^{2g+1}(x - a_j)$ (distinct roots) satisfy $D^2 = 4g + 4$.

Later it was shown that curves with $D^2 \leq 4g + 4$ turn out to be such hyperelliptic curves or a curve of [4;1].

Indeed, $D^2 = 16, g = 3$ by applying the theory of minimal models.

7.4. **Coolidge.** In 1928,Coolidge studied plane curves C and introduced the notion called the adjoint systems of special index j, which are defined to be $jK_S + D$, $j > 1$.

Coolidge:

If D is rational and $|2K_S+D| = \emptyset$, then D is tranformed into a line on $\mathbf{P}^2$ by a Cremona transformation

This looks like a Castelnuovo' criterion of rational surfaces which claims

S is a rational surface if and only if $P_2(S) = 0, \dim H^0(S, \Omega^1) = 0$.

In 1929 , the great(economic) recession happened.
During 30's , people around the world had hard times.

At that time, American geometers developed Cremonian geometry.
They studied properties of plane curves which are invariant under Cremona transformations.

Now, let me introduce a modern version (revival) of Cremonian geometry.

In Cremonian geometry, pairs (S, C) of nonsingular rational surfaces S and curves $C \subset S$ are objects of the study.

For pairs (S, C) where C is nonsingular, D stands for C.

8. ENCOUNTER WITH GENUS

In 1959, I was a highschool student, at the library for teachers , I read a book "Mathematics for citizens' edited by the Academy USSR and then looked at Riemann surfaces and genera, which seemed very beautiful.

I had a dream ; someday, I would study them and understand the properties of genera in detail.

Then, I became a university student and got friends with great interests of mathematics and theoretical physics, including Takuro Shintani, Kenich Yoshida, Motohiko Yoshimura.

When I was a junior, a famous professor Akizuki visited Ochanomizu women University.

They said boys could come into the women university and were able to listen to his lecture.

At he beginning, he said " You may know degree of curves, but Geschlecht is much more important ".

For example, curves of degree n have genus smaller than $\frac{(n-1)(n-2)}{2}+1$.

If it's nonsingular, the genus coincides with $\frac{(n-1)(n-2)}{2}$.

9. prof Kawada

Figure 5. kawada

As a senior, I was a member of prof Kawada's seminar.
The text is a lecture note on schemes by Dieudonne in English.
The theory of schemes were created by A.Grothendieck.

FIGURE 6. Grothendieck

When I was a graduate student, young bright professor appeared, whose name is Michio Kuga.

One day, he asked me " Do you know Fermat's last theorem in the case of polynomials'

Suddenly, I answered, " Use genus, the answer is immediate. If $n \geq 3$, then genus is positive. No rational solution."

I was confident that genera are important and decided to study them.

9.1. **case of algebraic curves.** If $g = 0$ then algebraic curves are rational curves; and the converse is true.

If $g = 1$ then algebraic curves turn out to be elliptic curves; and the converse is true.

9.2. **case of algebraic surfaces.** Algebraic surfaces have different kind of genera: geometric genus p_g, arithmetic genus p_a, linear genus p_l and so on.

Rational surfaces have $p_g = p_a = 0$. But the converse is not true.

There exit much more complicated genera like $P_2, P_3, P_4, P_6,$. Rational surfaces have $P_2 = p_a = 0$and the converse is true. Ruled surfaces have $P_4 = P_6 = 0$ and the converse is true.

For a point p of an algebraic surface, one can replace p by a line E. E is a projective line with $E^2 = -1$.

Such curves are called -1 curve, or an exceptional curve of the first kind. If S have no -1 curves, S becomes a minimal model. Indeed, in this case, a canonical divisor K_S is nef.I.e., intersection of K_S with any curve on S is non-negative.

Except for rational surfaces and ruled surfaces they have minimal models, which are central results of Zariski's theory of minimal models.

His theory have been successfully extended to the theory of complex analytic surfaces by Kodaira.

9.3. **Shafarevich.** When I was a graduate student, Kawada introduced a seminary note edited by students of Shafarevich such as Tyrin, Moishezohn.

It was a English translation from Russian text.
But the translator did not know mathematics at all.
He learned Russian while he stayed in a concentration camp after WW2.

In that seminar, from plurigenera, an invariant κ was introduced.

In the cases $\kappa = 0$ or $\kappa = 1$, structure of surfaces have been determined easily.
Then I read compact surfaces I,II,III by Kodaira.

9.4. **Kodaira and Spencer.** When I was a graduate student, Kodaira visited University of Tokyo in 1967. Next year he came back to Japan as professor of U. Tokyo.

FIGURE 7. 小平先生

9.5. **Deformation and plurigenera.** Joint work of Kodaira and Spencer on deformation of complex structures is very popular. Moreover,in the classification of complex surfaces, P_m play important roles.

Then Kodaira was professor of U.Tokyo and was as an assistant.

I asked Kodaira

“ Plurigenera P_m have to be invariant under deformation. Is this clear ?”

Kodaira replied.

“ I have never consider such problem. It may be true. ”

If $\kappa = 2$ and minimal, then $P_m = \frac{m(m-1)}{2}K^2 + 1 + p_a$ by Kodaira's vanishing theorem, which imply deformation invariance of plurigenera.

Using this and classificaton of algebraic surfaces, I proved that plurigenera are invariant under deformation.

On the other hand, Zarisiki developed the theory of divisors on an algebraic surface.

For a divisor D, one had complete linear systems $|mD|$ and the dimension $h^0(mD) = \dim |mD| + 1$.

He investigated $h^0(mD)$ as a function of m on a surface. In general, this is not a quadratic function(polynomial).

Meanwhile, I started to study $h^0(mD)$ for divisors D on algebraic varieties V of dimension n.

Theorem 1. *If there exists* $m_0 > 0$ *with* $h^0(m_0D) > 0$, *then there exists* m_1 *and* α, β *such that such that*

$$\alpha m^\kappa \leq h^0(mm_0D) \leq \beta m^\kappa$$

for $m > m_1$, *where* κ *is a nonnegative integer.*

Since κ depends on V and D, κ is written as $\kappa(V, D)$, which is called $D-$ dimension of V.

On the other hand, if $h^0(mD) = 0$ for any $m > 0$, then $\kappa(V, D) = -\infty$.

Applying this for $D = K_V$, we have canonical (divisor) dimension $\kappa(V, K_V)$, which is written simply $\kappa(V)$.

In this way,κ in Shafarevich seminar is generalized.

For $n-$ dimensional variety V, $\kappa(V) = -\infty, 0, 1, 2, \cdots, n$

Thus, varieties are classified into $n+2$ basic types.

If $0 < \kappa(V) < n$, there exist a fiber space $f : V^* \to W$ such that V^* is birationally equivalent to V,

W is an algebraic variety of dimension κ and general fibers $V_u = f^{-1}(u)$ have $\kappa(V_u) = 0$.

9.6. **birational classification of algebraic surfaces.** This result is a typical case of birational classification of algebraic surfaces S.

(1) If $\kappa(S) = -\infty$, then S birationally equivalent to either rational surfaces ($g = 0$) or irrational ruled surfaces ($g > 0$).
(2) If $\kappa(S) = 0$, then S is birationally equivalent to K3 surfaces , Enriques surfaces ($g = 0$) or abelian surfaces, hyperelliptic surfaces ($g = 1, 2$).
(3) If $\kappa(S) = 1$, then S is an elliptic surface of general type.
(4) If $\kappa(S) = 2$, then S is called a surface of general type.

The similar results for surfaces are expected to hold true for higher dimensional cases:

For example, if $f : V \to W$ is surjective and general fibers V_w are connected then the following inequalities expected to hold.

$$\kappa(V_w) + \dim W \geq \kappa(V) \geq \kappa(V_w) + \kappa(W).$$

The left hand inequality is easily verified. But The right hand seems hard to prove.

If V is birationally equivalent to an abelian variety then $q(V) =$ (the dimension of $H^0(V, \Omega_1)$) and $\kappa(V) = 0$.

In 1968 , I told the idea and basic results to Kodaira.

Kodaira said " I have never had such an idea, but your ideas seem OK."

In this way, the results of $\kappa(V, D)$ turns out to be my thesis.

At that time, everyday, around 5 o'clock, Kodaira knocked the door of my office, and said " It's time to go home. ".

Then we visited a coffee shop near the university and drank and had cakes, we talked about one hour every time.

The themes of his talk have wide range, including physics, science, future of our civilization, shortage of gas and so on.

On an evening, I said to him.

I wish to call Kodaira dimension for $\kappa(V)$ instead of canonical dimension, because the notion started from Kodaira's work.

Kodaira smiled and said nothing.

10. prof Atiyah

Suggested by Kodaira, I would go to IAS at Princeton as temporary member.

So I sent a letter to prof Atiyah with the research project, in which Kodaira dimension plays central role.

In replying letter, Atiyah told me that Prof Moishezohn has the same idea as yours.

I was shocked , because Moishezohn is an excellent mathematician and one of authors of Shafarevich seminary note.

Later ,classification theory of higher dimensional algebraic varieties developed with the work by K.Ueno, Y.Miyaoka,T.Fujita,S.Mori,Y.Kaw
M.Reid,Viehweg, J.Kollar and so on.

Including the deformation invariance of $\kappa(V)$, the most parts of the expected results have been verified thanks to the efforts of many mathematicians.

On the other hand, Moishezohn succeeded in leaving USSR and I met him at Tokyo.

We had enough time to talk on politics and dayly lives of citizens in USSR.

He got a position in Israel and then became a professor of University of Utah.

Suddenly he died of illness.

In 1979, Miles Reid gave me a letter of Shafarevich, in which Shafarevich inserted a message to Western countries.

He claims in that message, under the Soviet political system, mathematicians had very hard times.

Reid told me the message should be published in some Japanese popular magazine. So I tried and finally the very popular magazine BungeiShunju agreed to publish it as soon as possible.

After the magazine was published, I sent a letter to Shafarevich. But no reply at all.

10.1. **Wikipedia.** Later on, the concept of Kodaira dimension becomes very popular.

For instance, in the **Wikipedia**:

In algebraic geometry, the Kodaira dimension $\kappa(X)$ *measures the size of the canonical model of a projective variety* X*.*

Kodaira dimension is named for Kunihiko Kodaira.

The name and the notation κ *were introduced by Igor Shafarevich in the seminar Shafarevich 1965.*

In 1995, Shafarevich was invited to be a keynote speaker of the second Asian Congress of Mathematics.

This is the first occasion to meet him. He thanked me for the letter that I sent before.

In his keynote speech, he mentioned iitaka variety and Kodaira dimension.

Proceedings of the Second Asian Mathematical Conference, 1995

ON SOME ARITHMETIC PROPERTIES OF ALGEBRAIC VARIETIES

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1. Introduction

I will try to give a short survey of what is called Arithmetics of "Algebraic Varieties" — its main results and problems.

Let $X \subset \mathbb{P}^N$ be an algebraic variety defined in the projective space $\mathbb{P}^N$ by equations

This description can be easily generalised to algebraic varieties of arbitrary dimension n. Namely, we consider rational differential forms of type $f(du_1 \wedge \ldots \wedge du_n)^m$ for some fixed system of algebraically independent rational functions $u_1, \ldots, u_n$ on X, a rational function f and $m \geq 1$. All regular forms (i.e. having no poles on X) for fixed weight m form a finite dimensional vector space Ω_m and in the same way as before we obtain a rational mapping $\varphi_m : X \to \mathbb{P}(\Omega_m)$ into a projective space. It can be proved that for all m sufficiently large and divisible by a fixed integer, the (closure of the) varieties $\varphi_m(X)$ are birationally isomorphic. So (up to birational isomorphism) there exists a single variety $I(X)$ isomorphic to all these $\varphi_m(X)$, which is called the Iitaka variety of X, and a single mapping

$$\varphi : X \to I(X) \tag{3}$$

of X onto $I(X)$. Of course, if all $\Omega_m = 0$ neither the Iitaka variety nor the mapping φ are defined. The dimension κ of the variety $I(X)$ is called the Kodaira dimension of X. If all $\Omega_m = 0$ for $m \geq 1$ and $I(X)$ is not defined, we set $\kappa = -\infty$. So κ can take the $n+2$ values $\kappa = -\infty, 0, 1, \ldots, n$.

(In this short survey we completely ignore the difficulties which arise in connection with the fact that the variety $I(X)$ and the fibres of the mapping φ may have singular points even when X has none. These difficulties are overcome in cases $n = 2$ and $n = 3$ and there exists a program of resolving them in the general case, known as "Mori's program".)

11. Logarithmic Kodaira dimension

In 1971–72 I stayed at IAS and started the research of varieties whose universal cover is the $n-$ dimensional affine space.

To study the ramification of threefolds, I introduced the notion of logarithmic Kodaira dimension of quasi–projective varieties..

Moreover, logarithmic plurigenera and strict rational maps are introduced.

Actually, given nonsingular quasi-projective variety V, one has projective variety $\overline{V}$ and a divisor D on $\overline{V}$ such that the complement $\overline{V} - D$ is V.

Take a nonsingular $\overline{V}^*$ and a birational morphism $\mu : \overline{V}^* \to \overline{V}$ such that the total inverse image $D^* = \mu^{-1}(D)$ is a divisor of simple normal crossings.

Then it was proved that $H^0(m(K_{\overline{V}^*} + D^*))$ depends upon only V, independent of $\overline{V}^*$ and D^*.

The dimension $H^0(m(K_{\overline{V}^*} + D^*))$ is called the logarithmic plurigenera of V.

In this way, a new birational geometry was born;

it is the revival of the classical birational geometry of projective varieties

The new geometry is successfully applied to the study of affine varieties.

However, if D is irreducible, one may consider the proper inverse image $D^\sharp = \mu^{-1}[D]$ instead of the total inverse image $\mu^{-1}(D)$.

Then the complement of $D^\sharp$ is not proper birationally equivalent to V.

However, the pairs $(\overline{V}^*, D^*)$ are birationally equivalent to $(\overline{V}, D)$

This is also anther development of birational geometry, that is the geometric study of DVR(discrete valuation ring) instead of function fields.

If it is 2– dimensional, this turns out to be birational geometry of pairs of algebraic curves $C$and surfaces$S$ such that $C \subset S$.

If surfaces are rational, this is birational geometry of algebraic plane curves.

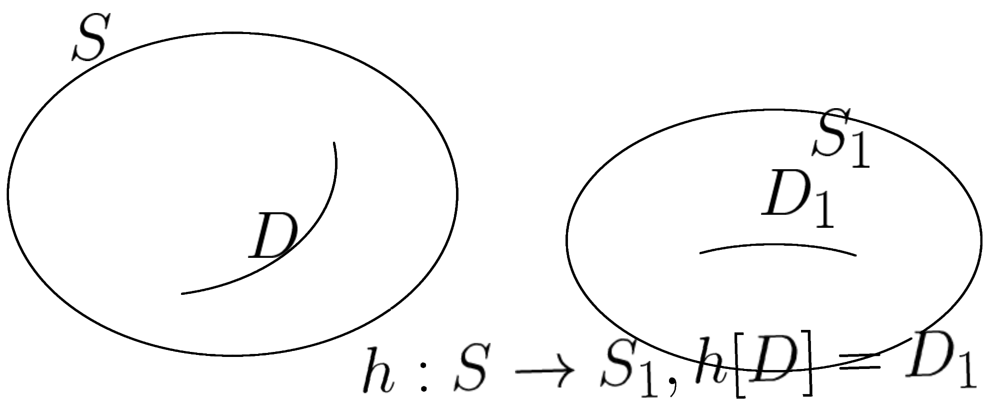


FIGURE 10. 双有理同値

Given a nonsingular projective algebraic rational surface S and a curve C on S , we have a notion of birational equivalence.

Two pairs (S, C) and (S_1, C_1) are birationally equivalent, if there exists birational map $h : S \to S_1$ such that the proper transform $h[C]$ of C coincides with C_1. Then they are called birationally equivalent pairs.

If C is nonsingular, we use D instead of C .

Birational geometry of such pairs is called Cremonian geometry.

Given (S, D), one has mixed plurigenera which is defined to be $P_{ma}[D] = \dim |mK_S + aD| + 1$.

These are invariant under birational equivalence.

$P_{m,m}[D]$ turns out to be logarithmic m genera $\overline{P_m}(S - D)$ of an open surface . $S - D$.

Moreover, the Kodaira dimension of the pair is defined to be $\kappa[D] = \overline{\kappa}(S - D)$.

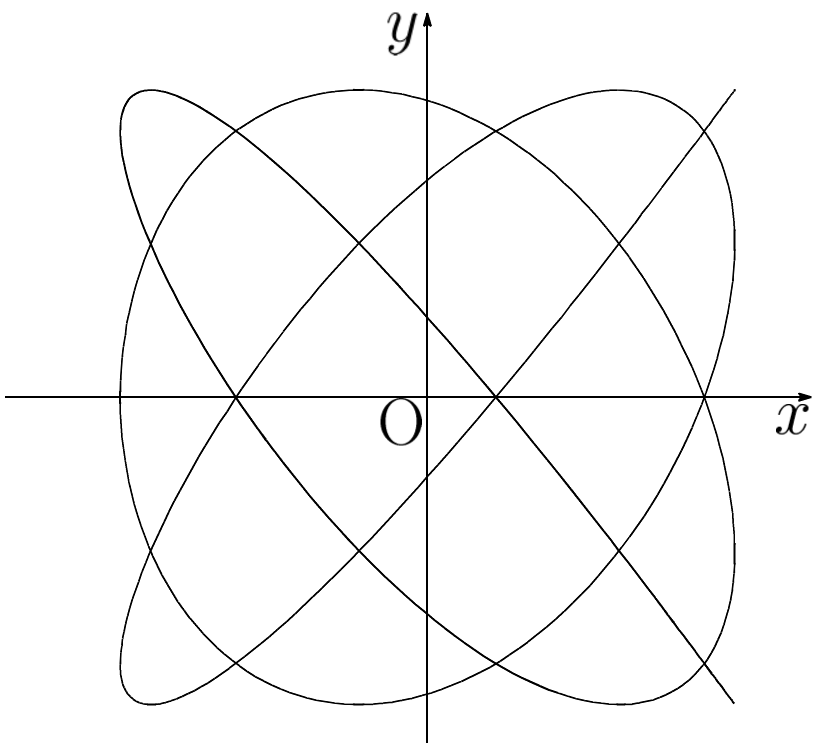
Thus $\kappa[D] = \kappa(S, Z)$, where $Z = K_S + D$.

Since S is rational, it follows that $P_{1,1}[D] = g(D)$. Here $g(D)$ is the genus of D.

Here is a classification of pairs when $\kappa[D] < 2$.
$\kappa[D]$ is defined to be $Z-$ dimension, whrere $Z = K_s + D$.

- If $\kappa[D] = -\infty$ then (S, D) is birationally equivalent to ($\mathbf{P}^2$,Line).
- If $\kappa[D] = 0$ then (S, D) is birationally equivalent to either ($\mathbf{P}^2$,C_3), C_3 being a nonsingular cubic or ($\mathbf{P}^2$,C_6), C_6 being a sextic curve with 10 double point.
- If $\kappa[D] = 1$ then (S, D) is birationally equivalent to
  1) ($\mathbf{P}^2$,C_{3m}), C_3 being a curve of degree $3m$ with nine $m-$ ple points or
  2) ($\mathbf{P}^2$,C'_{3m}), C'_3 being a curve of degree $3m$ with nine $m-$ ple points and a double point,
  3) ($\mathbf{P}^2$,H_g), H_g being a hyperellitptic curve defined by $y^2 - (x - a_1)(x - a_2)(x - a_{2g+2}) = 0$, $g \geq 2$, where a_j' are mutually different to each other.

Hence, hereafter assume that $\kappa[D] = 2$.



The curve defined by $x = \cos(6t), y = \cos(7t)$ is a rational curve of degree 7, which has 15 double points. This is the simplest example of rational curves with $\kappa[D] = 2$

If (S, D) is relatively minimal, it's minimal.

So it suffices to study structures of minimal models (S, D).

Theorem 2. *Suppose that $\sigma \geq 7$. Then*

- $P_{2,1}[D] = Z^2 - \overline{g} + 1 = A + 1.$
- $P_{3,1}[D] = 3A - \alpha + 1 = \Omega - \omega + 1.$

Here, $\overline{g} = g - 1$, $A = Z^2 - \overline{g}$; $\alpha = 4\overline{g} - D^2$,
$\Omega = (3Z - 2D) \cdot Z = 3Z^2 - 4\overline{g}$. Moreover, $\omega = 3\overline{g} - D^2$.

12. $\#-$ MINIMALITY

Minimal pairs are obtained from some kind of singular models, namely, $\#$ minimal pairs which will be defined below.

Any nontrivial $\mathbf{P}^1-$ bundle over $\mathbf{P}^1$ has a section Δ_∞ with negative self intersection number, which is denoted by a symbol Σ_B, where $-B = {\Delta_\infty}^2$ if $B > 0$. Σ_B is said to be a Hirzebruch surface of degree B after Kodaira.

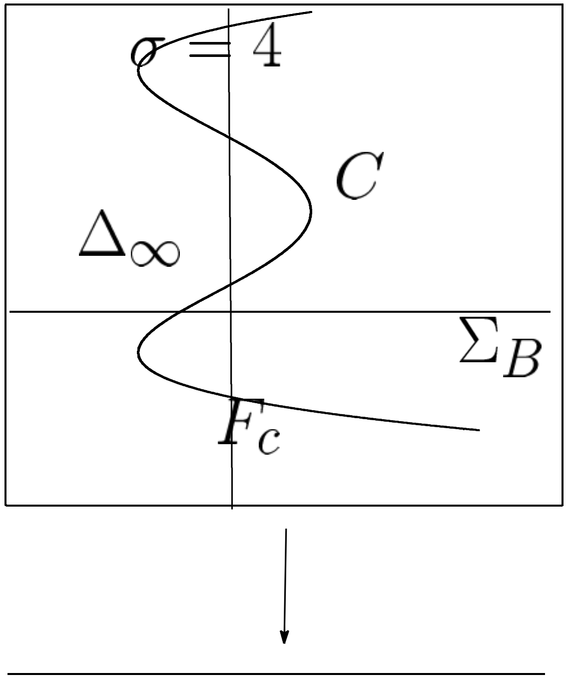
Let Σ_0 denote the product of two projective lines.

The Picard group of Σ_B is generated by a section Δ_∞ and a fiber $F_c = \rho^{-1}(c)$ of the $\mathbf{P}^1-$ bundle, where $c \in \mathbf{P}^1$ and $\rho : \Sigma_B \to \mathbf{P}^1$ is the projection.

Let C be an irreducible curve on Σ_B. Then $C \sim \sigma\Delta_\infty + eF_c$, for some σ and e. Here the symbol $\sim$ means the linear equivalence between divisors. We have $C \cdot F_c = \sigma$ and $C \cdot \Delta_\infty = e - B \cdot \sigma$. Note that $\kappa[\Delta_\infty] = -\infty$.

Hereafter, suppose that $C \neq \Delta_\infty$. Thus $C \cdot \Delta_\infty = e - B \cdot \sigma \geq 0$ and hence, $e \geq B\sigma$.

By $\nu_1, \nu_2, \cdots, \nu_r$ we denote the multiplicities of all singular points (including infinitely near singular points) of C where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$.



The symbol $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$ is said to be the **type** of (Σ_B, C).

Definition 1. *the pair* (Σ_B, C) *is said to be* # ***minimal*** *, if*

- $\sigma \geq 2\nu_1$ *and* $e - \sigma \geq B\nu_1$;
- *moreover, if* $B = 1$ *and* $r = 0$ *then assume* $e - \sigma > 1$.

Using elementary transformations, we get

Theorem 3. *If* D *is not transformed into a line on* $\mathbf{P}^2$ *by Cremona transformations, then* $\kappa[D] \geq 0$.
Minimal pair (S, D) *is obtained from a* # *minimal pair* (Σ_B, C) *by shortest resolution of singularities of* C *using blowing ups except for* $(S, D) = (\mathbf{P}^2, C_d)$, C_d *being a nonsingular curve.*

13. BASIC RESULTS

Introduce Y by $Y = \sum_{j=1}^{r} \nu_j$ and X by $X = \sum_{j=1}^{r} {\nu_j}^2$. Moreover, defining $w = 4 - \delta_{1B}$,we get $w = 4$ if $B \neq 1$. $w = 3$, otherwise.

Let k denote $wp + 2u$.

Proposition 1. *Suppose that $B \leq 2$. Then*
(1) $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
(2) $Y = 8\nu_1 + k + \omega_1$.

Here $\tilde{k} = kp - 2p^2, \omega_1 = \omega - \overline{g}$.

14. INVARIANTS OF THREE GENERATIONS

Birational invariants are classified into three generations.

TABLE 1. three generations

1st generation	$e, \nu_1, \nu_2, \cdots, \nu_r, \sigma, B, d$
	$k = (4 - \delta_{B,1})p + 2u, \tilde{k} = p(k - 2p)$
2nd generation	$g =$ genus(Riemann),
3rd generation	four fundamental invariants $\alpha, \omega, A, \Omega$
	$P_{2,1}[D]$(Coolidge,1928), $P_{3,1}[D]$, $P_{m,a}[D]$,

$$\alpha = (D + 2K) \cdot D, \omega = \frac{(D + 3K) \cdot D}{2}, A = \frac{(D + 2K) \cdot Z}{2},$$

$$\Omega = (D + 3K) \cdot Z$$

$Z = D + K_S, K_S :$ canonical divisors

Note that genus was defined by Riemann.

$$g - 1 = \frac{(D + K) \cdot D}{2} = \frac{Z \cdot D}{2}.$$

dogma (dogmatic opinion)

Each fundamental invariant of 3rd generation determines birational invariants.

14.1. **pairs with** $\omega = 1, 2$.

Under the assumption $\sigma \geq 7$, we show the list of types of pairs with $\omega = 1, 2, 3, 4, 5, 6$. However, associated types are omitted, for simplicity.

TABLE 2. $\omega = 1, 2$

ω	σ	type	genus
1	7	$[7 * 9, 1; 1]$	27
2	7	$[7 * 9, 1; 2]$	26
2	8	$[8 * 8; 4^7]$	7
2	8	$[8 * 8; 4^7, 3]$	4
2	8	$[8 * 8; 4^7, 3^2]$	1
2	10	$[10 * 11; 5^9]$	0
2	12	$[12 * 12; 6^7, 5, 4]$	0

14.2. **pairs with $\omega = 3$.**

Table 3. $\omega = 3$

ω	σ	type	genus
3	7	$[7 * 9, 1; 2^2]$	25
3	8	$[8 * 9; 4^9]$	2
3	8	$[8 * 8; 4^7, 2]$	6
3	8	$[8 * 8; 4^7, 3, 2]$	3
3	8	$[8 * 8; 4^7, 3^2, 2]$	0
3	10	$[10 * 10; 5^7, 4, 3]$	2
3	10	$[10 * 10; 5^7, 4]$	5
3	12	$[12 * 12; 6^6, 5^3]$	1
3	14	$[14 * 14; 7^7, 6, 4]$	1
3	15	$[15 * 22, 1; 7^9]$	0
3	16	$[16 * 16; 8^6, 7^2, 6]$	0
3	20	$[20 * 20; 10^7, 9, 5]$	0

Assume that $\sigma \geq 7, \nu_1 \geq 3$.

fundamental problems

(1) Enumerate types for given birational invariants of 3rd generation

(2) relationships between birational invariants of 3rd generation and those of 2nd generation

 (a) relationships between ω and g.

 (b) relationships between A and g.

(3) relationships between birational invariants of 3rd generation and those of 1st generation

 (a) relationships between ω and k; ω and σ;

 (b) relationships between A and k; A and σ;

examples obtained so far.

(1) The first inequalities $\sigma \leq (\alpha+3)(\alpha+2)$ (By O.Matsuda).

(2) If the first equalities $\sigma = (\omega+1)(\omega+2)$ fail, prove $\sigma \leq \omega^2+\omega+1$.

(3) The first inequalities in two variables $\sigma \leq \omega_1^2+\omega_1+2+2\overline{g}$(1122 inequality) .

14.3. **curves of degree 5.** Plane curves C of degree 5 are classified into the following four types according to $\kappa[D]$ and g.

- $\kappa = 2$
  (1) $g = 6$. Then C is a non-singular plane curves.
  (2) $g = 5$. Then C has a double point. The type is $[3*5, 1; 1]$.
  (3) $g = 4$. Then C has 2 double points. The type is $[3*3; 1]$.
  (4) $g = 3$. Then C has 3 double points. The type is $[3*4, 1; 1]$.
- $\kappa = 1$
  (1) $g = 3$. C has a triple point and $\kappa = 1$, the type $[2*5, 1; 1]$.
  (2) $g = 2$. C has a triple point and a double point with $\kappa = 1$, the type $[2*3, 1; 1]$.
- $\kappa = 0$ Then $g = 1$ and by Cremona transformation the curves are transformed into a non-singular cubic.
- $\kappa = -\infty$. Then $g = 0$ and By Cremona transformation the curves are transformed into a line.

15. SECOND INEQUALITIES

Assume that $\sigma \geq 7$.

Theorem 4. (1) $\sigma \leq (\omega+1)(\omega+2)$.

(2) *If* $\sigma = (\omega+1)(\omega+2)$ *,then* $[2\nu_1 * 2\nu_1; \nu_1^7, \nu_1 - 1, \nu_r]$*, Here,* $\nu_1 = \frac{\nu_r(\nu_r - 1)}{2}$ *and* $\omega = \nu_r - 2$.

(3) *If* $\sigma < (\omega+1)(\omega+2)$*,then* $\sigma \leq \omega(\omega+1)+2$ *except for the following;*

(a) $(\omega = 2)$, $[10 * 11; 5^9]$;

(b) $(\omega = 3)$, $[15 * 22, 1; 7^9]$ *and* $[16 * 16; 8^6, 7^2, 6]$;

(4) *If* $\sigma < \omega(\omega+1)+2$ *then* $\sigma \leq \omega(\omega-1)+4$ *except for the following;*

(a) $(\omega = 3, g = 1)$, $[12 * 12; 6^6, 5]$;

(b) $(\omega = 4, g = 1)$, $[18 * 18; 9^7, 7, 6]$;

(c) $(\omega = 4, g = 0)$, $[19 * 19; 9^9]$;

(d) $(\omega = 4g = 0)$, $[20 * 20; 10^5, 9^3, 8]$.

Theorem 5. (1) $\sigma \leq (\alpha+3)(\alpha+2)$ *(By O.Matsuda);*

(2) *If* $\sigma = (\alpha+3)(\alpha+2)$ *then* , $[2\nu_1 * 2\nu_1; \nu_1^7, \nu_1 - 1, \nu_r]$,

Here,$\nu_1 = \frac{\nu_r(\nu_r-1)}{2}$ *and* $\omega = \nu_r - 2$.

(3) $\sigma \leq \alpha^2 + (1-4\overline{g})\alpha + 4\overline{g}^2 + 2$;

(4) *If* $g > 0$ *then* $\sigma \leq \alpha(\alpha+1)+2$;

(5) *IF* $\sigma < (\alpha+3)(\alpha+2)$ *then* $\sigma \leq \alpha(\alpha+1)+2$,*except for the following;*

(a) $(\alpha = 1)$, $[10 * 11; 5^9]$;

(b) $(\alpha = 2)$, $[15 * 22, 1; 7^9]$, $[16 * 16; 8^6, 7^2, 6]$;

(c) $(\alpha = 3)$, $[19 * 19; 9^9]$, $[19 * 38, 2; 9^9]$, $[20 * 20; 10^5, 9^3, 8]$, $[22 * 22; 11^6, 10^2, 7]$, $[22 * 22; 11^7, 8^2]$;

(d) $(\alpha = 4)$, $[23 * 35, 1; 11^9]$, $[24 * 24; 12^4, 11^4, 10]$, $[24 * 25; 12^7, 10^2]$, $[25*37, 1; 12^8, 9]$, $[28*29; 14^8, 8]$, $[30*30; 15^7, 13, 8]$;

(e) $(\alpha = 5)$, $[36 * 37; 18^8, 9]$, $[38 * 38; 19^7, 17, 9]$;

(f) $(\alpha = 6)$, $[46 * 46; 23^6, 22^2, 10]$.

In the case when $B \leq 2$, we put $\varepsilon_B = 1 + \frac{B}{2}$.

Theorem 6. *If* $\sigma \geq 7, \nu_1 \geq 3$ *then*

$$e \leq \varepsilon_B \cdot (\omega + 1)(\omega + 2), \tag{1}$$

$$e \leq \varepsilon_B \cdot (A + 3)(A + 2). \tag{2}$$

Given ω or A, e is bounded. Thus there are a finite number of types of $\#-$ minimal pairs.

$X = \sum_{j=1}^{r} \nu_j^2, Y = \sum_{j=1}^{r} \nu_j, \widetilde{\mathcal{Z}} = \nu_1 Y - X.$

Proposition 2. *If $B \leq 2$ then*

- $X = 8\nu_1^2 + 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g},$
- $Y = 8\nu_1 + k + \omega_1.$

Here, $\tilde{k} = kp - 2p^2, \omega_1 = \omega - \overline{g}.$

16. ESTIMATE OF k IN TERMS OF ω

Assume $\sigma \geq 7$, $\nu_1 \geq 3$.
If $B \geq 3$ then $\omega - k \geq \sigma - 1$.
Thus $\omega + 1 \geq \sigma + k$.
$\sigma + k$ is bounded by ω.
Since $\sigma \geq 7$, it follows that $\omega - k \geq 6$.
When$\omega - k < 7$, one can suppose that $B \leq 2$.

Proposition 3. $k \leq \omega$, *If* $g > 0$ *then* $k \leq \omega - 1$.

16.1. **The case when $\omega - k = 0$.**

If $\omega - k = 0$ then $g = 0$ and the types are as follows:

- I) If $p = 0$ then $[10 * 11; 5^9]$. $k = 2$.
- II) If $p = 1$, then
  1) $[(15 + 8u) * (22 + 13u), 1; (7 + 4u)^9]$, $(k = 3, 5, 7, 9, \cdots)$
  2) $[(19 + 8u) * (19 + 9u); (9 + 4u)^9]$, Here $u \geq 0.(k = 4, 6, 8, 10, \cdots)$
- III) If $p > 1$ then $p = 2$ and moreover, $[28 * 41, 1; 13^9]$. $\omega = k = 6$.

16.2. **The case when $\omega = k+1$.**

Assume $i = \omega - k = 1$.

II.

(1) $[(9+4u)*(13+7u),1;(4+2u)^{10}]$, where $\omega = 4+2u, g = 0$
, $k = 3+2u$.

(2) $[(11+4u)*(11+5u);(5+2u)^{10}]$, where $\omega = 5+2u, g = 0$
and $k = 4+2u$.

Here, $p = 1; \omega = k{-}1 = 4, 6, 8, 10, \cdots, 5, 7, 9, 11, \cdots, 7, 25, 26, 33, \cdots$.

III.

$g = 0$.

1). equimultiple.

(1) $[109 * 166, 1; 52^9]$, where $\omega = 26, k = 25$.
(2) $[121 * 177, 1; 56^9]$. where $\omega = 28, k = 27$.
(3) $[136 * 212, 1; 66^9]$. where $\omega = 33, k = 32, Z^2 = 31$.
(4) $[136 * 144; 66^9]$. where $\omega = 33, k = 32, Z^2 = 31$.
(5) $[106 * 106; 50^9]$. where $\omega = 25, k = 24$.
(6) $[106 * 159, 1; 50^9]$. where $\omega = 25, k = 24$.
(7) $[16 * 23, 1; 7^{10}]$. where $\omega = 7, k = 6$.

$\nu_1 - 1$

(1) $[40 * 40; 19^8, 18]$. , where $\omega = 9, k = 8$.
(2) $[40 * 60, 1; 19^8, 18]$. , where $\omega = 9, k = 8$.
(3) $[45 * 66, 1; 21^8, 20]$. , where $\omega = 10, k = 9$.

Here, $p \geq 2$.

2). The case when $g = 1$

(1) $[36 * 36; 17^9]$., where $\omega = 9, k = 8$.

(2) $[41 * 60, 1; 19^9]$., where $\omega = 10, k = 9$.

(3) $[36 * 54, 1; 17^9]$., where $\omega = 9, k = 8$.

3). The case when $g = 2$

(1) $[8 * 9; 4^9]$. , where $\omega = 3, k = 2$.

17. INVARIANT $\widetilde{\mathcal{Z}}$

Let $\nu_1 Y - X$ be denoted by $\widetilde{\mathcal{Z}}$.

Then $\widetilde{\mathcal{Z}} = \nu_1 Y - X = \sum_{j=2}^{\nu_1 - 1} (\nu_1 - j) j t_j \geq 0$, t_j being the number of $j-$ ple singular points on C.

(3) $$0 \leq \widetilde{\mathcal{Z}} = \nu_1(\omega - \overline{g} - k) - \tilde{k} - \omega_1 + 2\overline{g}.$$

17.1. **invariant $\mathcal{Z}^*$.** Hereafter suppose that $\nu_1 \geq 3$.

Introducing invariants $\overline{\nu_j}$ and $\overline{Y}$ by $\overline{\nu_j} = \nu_j - 1$ and $\overline{Y} = \sum_{j=1}^{r} \overline{\nu_j}$, respectively, we obtain $\overline{Y} = Y - r$ and

$$\overline{Y} = 8\mu + k + A_1.$$

Moreover, introduce an invariant $\overline{X}$ by

$$\overline{X} = \sum_{j=1}^{r} \overline{\nu_j}^2 = X - 2Y + r,$$

which satisfies that

$$\overline{X} = 8\mu^2 + 2k\mu + \tilde{k} - A_1 - 2\overline{g}.$$

Here, for simplicity, let μ stand ν_1. Thus $\mu \geq 2$.

17.2. **case when $B > 2$.** However, if $B > 2$, we have fundamental equalities :

(1) $\overline{Y} = B_2\sigma + 8\mu + k + A_1$,

(2) $\overline{X} = B_2\sigma(\sigma - 2) + 8\mu^2 + 2k\mu + \tilde{k} - A_1 - 2\overline{g}$,

where $B_2 = B - 2$ for $B \geq 2$.

Moreover, if $B \leq 2$, we put $B_2 = 0$.

Defining an invariant $\mathcal{Z}^*$ to be $\mu\overline{Y} - \overline{X}$, we obtain

$$-\mu k - \tilde{k} + \nu_1 A_1 + 2\overline{g} - B_2\sigma(\sigma - 2 - \mu) = \mathcal{Z}^*$$

and

$$\mathcal{Z}^* = \sum_{j=2}^{\nu_1 - 1} (\nu_1 - j)(j - 1)t_j.$$

$$\mathcal{Z}^* = (\mu - 1)y_1 + 2(\mu - 2)y_2 + 3(\mu - 3)y_3 + \ldots \geq 0,$$

where $y_1 = t_2 + t_\mu, y_2 = t_3 + t_{\mu-1}, y_3 = t_4 + t_{\mu-2}, \cdots$.

Moreover, we get

$$B_2\sigma(\sigma - 2 - \mu) \leq -\mu k - \tilde{k} + \nu_1 A_1 + 2\overline{g}.$$

Proposition 4. *If* $B \geq 3,$ *then*

$$\sigma(\sigma - 2 - \mu) \leq \nu_1 A_1 + 2\overline{g} - \mu k - \tilde{k}. \tag{4}$$

Suppose that $p > 0$. Then $\sigma - 2 - \mu \geq 1 + 2(\mu + 1) - 2 - \mu = 1 + \mu$ and $\tilde{k} - k \geq -2$. Hence,

$$\sigma(\mu + 1) \leq \sigma(\sigma - 2 - \mu) \leq \nu_1 A_1 + 2\overline{g} - \mu k - \tilde{k}.$$

However,

$$\nu_1 A_1 + 2\overline{g} - \mu k - \tilde{k} = (\mu+1)A - (\mu+1)k + (1-\mu)\overline{g} - \tilde{q},$$

where $\tilde{q} = \tilde{k} - k \geq -2$.

Therefore

$$\sigma(\mu+1) \leq (\mu+1)(A-k) + (1-\mu)\overline{g} + 2.$$

If $\overline{g} \geq 0$, then $(\sigma - (A-k))(\mu+1) \leq 2$. Hence, $\sigma \leq A - k$.

If $\overline{g} = -1$, then

$$\sigma(\mu+1) \leq (\mu+1)(A-k) + \mu + 1 = (\mu+1)(A-k+1)$$

Hence, $\sigma \leq A - k + 1$.

By $\mu \geq 2$, we get $\sigma \geq 7$ and so $k + 6 \leq A$. Thus we obtain the next result.

Proposition 5. *If $B \geq 3$ and $p > 0$, then $\sigma + k - 1 \leq A$. In particular, $k + 6 \leq A$.*

18. Hartshorne's identities

Suggested by Hartshorne, consider a divisor $2D + \sigma K_S$. The intersection numbers of this and divisor$D$ and Z, will produce useful identities among invariants.

By $\tilde{\theta}_2$ we denote $(2C + \sigma K_0) \cdot C$. From $(2D + \sigma K_S) \cdot D = 2\sigma\overline{g} - (\sigma - 2)D^2$ it follows that

$$2\sigma\overline{g} - (\sigma - 2)D^2 = \tilde{\theta}_2 + pY + 2\tilde{\mathcal{Z}}.$$

By the way,

$$\begin{aligned}
\tilde{\theta}_2 &= (2C + \sigma K_0) \cdot C \\
&= (\sigma Z_0 - (\sigma - 2)C) \cdot C \\
&= 2\sigma\overline{g}_0 - (\sigma - 2)C^2 \\
&= \sigma(\sigma\widetilde{B} - 2\sigma - \tilde{B}) - (\sigma - 2)\sigma\widetilde{B} \\
&= \sigma(\widetilde{B} - 2\sigma).
\end{aligned}$$

If $B \geq 2$ then $\widetilde{B} - 2\sigma = 2u + (B-2)\sigma \geq 0$. In particular, if $B \geq 3$ then $\widetilde{B} - 2\sigma \geq \sigma$.

If $B = 0$ then $\widetilde{B} - 2\sigma = 2u \geq 0$.

However, if $B = 1$ then $\widetilde{B} - 2\sigma = 2e - 3\sigma$.

In the case when $2e - 3\sigma \geq 0$, then $\tilde{\theta}_2 \geq 0$.

In the case when $2e - 3\sigma < 0$, letting $L = -(2e - 3\sigma) > 0$, we consider $\tilde{\theta}_3 = (3C + eK_0) \cdot C$.

Then

$$\begin{aligned}
\tilde{\theta}_3 &= (3C + eK_0) \cdot C \\
&= (eZ_0 - (e-3)C) \cdot C \\
&= 2e\overline{g}_0 - (e-3)C^2 \\
&= e(\sigma\widetilde{B} - 2\sigma - \widetilde{B}) - (e-3)\sigma\widetilde{B} \\
&= -2\sigma e - e\widetilde{B} + 3\sigma\widetilde{B} \\
&= -2\sigma e - 2e^2 + e\sigma + 3\sigma(2e - \sigma) \\
&= \sigma(2e - 3\sigma) - e(2e - 3\sigma) \\
&= L(e - \sigma) \\
&= L(u + \nu_1).
\end{aligned}$$

Thus $\tilde{\theta}_3 = L(u + \nu_1) \geq 3L > 0$.

Moreover,

$$2e\overline{g} - (e-3)D^2 = \tilde{\theta_3} + (p+u)Y + 3\tilde{\mathcal{Z}}.$$

We say that the sign of the type (S, D) is $\mathrm{RH}_{(+)}$ if either $B \neq 1$ or $B = 1$ and $2e - 3\sigma \geq 0$.

Otherwise, we say that the sign of the type is $\mathrm{RH}_{(-)}$, namely in the case when $B = 1$ and $2e - 3\sigma < 0$.

Instead of D, we take Z. Then $(2D + \sigma K_S) \cdot Z = \sigma Z^2 - 2\overline{g}(\sigma - 2)$ and by ${\theta_2}^*$ we denote $(2C + \sigma K_0) \cdot Z_0$.

Thus

$$\sigma Z^2 - 2\overline{g}(\sigma - 2) = {\theta_2}^* + p\overline{Y} + 2\mathcal{Z}^*.$$

We want to show that ${\theta_2}^* \geq 0$ in the case of $\mathrm{RH}_{(+)}$. Actually,

$$\begin{aligned}
{\theta_2}^* &= (2C + \sigma K_0) \cdot Z_0 \\
&= (\sigma Z_0 - (\sigma - 2)C) \cdot Z_0 \\
&= \sigma Z_0^2 - 2\overline{g_0}(\sigma - 2) \\
&= \sigma(\sigma \widetilde{B} - 4\sigma - 2\widetilde{B} + 8) - (\sigma - 2)(\sigma \widetilde{B} - 2\sigma - \widetilde{B} + 2) \\
&= (\sigma - 2)(\widetilde{B} - 2\sigma)
\end{aligned}$$

Since ${\theta_2}^* = (\sigma - 2)(e - B\sigma + 2 - 2\sigma) = (\sigma - 2)(2u + B_2\sigma)$, it follows that ${\theta_2}^* = (\sigma - 2)(2u + B_2\sigma) \geq 8u \geq 0$, if $B \neq 1$.

If $B = 1$ then ${\theta_2}^* = (\sigma - 2)(2e - 3\sigma)$.

Hence, if $2e - 3\sigma \geq 0$, then ${\theta_2}^* \geq 4(2e - 3\sigma) \geq 0$.

In the case when the sign of the type is $\mathrm{RH}_{(-)}$, namely if $2e - 3\sigma = -L$ is negative, consider $(3C + eK_0) \cdot Z_0$, which we denote by ${\theta_3}^*$.

Then

$$eZ^2 - 2(e-3)\overline{g} = \theta_3{}^* + (p+u)\overline{Y} + 3\mathcal{Z}^*.$$

By the way,

$$\begin{aligned}
\theta_3{}^* &= (3C + eK_0) \cdot Z_0 \\
&= (eZ_0 - (e-3))C \cdot Z_0 \\
&= eZ_0^2 - 2\overline{g_0}(e-3) \\
&= e(\sigma\widetilde{B} - 4\sigma - 2\widetilde{B} + 8) - (e-3)(\sigma\widetilde{B} - 2\sigma - \widetilde{B}) \\
&= e(2 + 5\sigma - 2e) - 3\sigma(\sigma+1) \\
&= e(2 + 2\sigma + L) - 3\sigma(\sigma+1) \\
&= 2e(\sigma+1) - 3\sigma(\sigma+1) + eL \\
&= (\sigma+1)(2e - 3\sigma) + eL \\
&= L(e - \sigma - 1) \\
&= L(u + \nu_1 - 1) \geq 2L(u+2) > 0.
\end{aligned}$$

Dear Shigeru-san,

Nice to hear from you.

I have retired from teaching, but still do mathematics and travel to conferences.

I am curious about "Hartshorne's identities".

Can you point me to which paper they are in, on what page?

i found many papers on your website.

With best wishes,

Robin Hartshorne

Since $\omega = 3\overline{g} - D^2$, it follows that

$$2\sigma\overline{g} - (\sigma - 2)D^2 = 2\sigma\overline{g} + (\sigma - 2)(\omega - 3\overline{g})$$
$$= (\sigma - 2)\omega - (\sigma - 6)\overline{g}.$$

Therefore,

$(2D + \sigma K_S) \cdot D = (\sigma - 2)\omega - (\sigma - 6)\overline{g}$ and it follows that

(5) $$(\sigma - 2)\omega - (\sigma - 6)\overline{g} = \tilde{\theta}_2 + pY + 2\tilde{\mathcal{Z}}.$$

$\tilde{\theta}_2 + pY + 2\tilde{\mathcal{Z}}$ is denoted by $\tilde{\Theta}_2$. Thus, when the signature of type is $\mathrm{RH}_{(+)}$, then

$$(\sigma - 2)\omega \geq (\sigma - 6)\overline{g}.$$

Hence, if $\sigma \geq 8, \nu_1 \geq 3$ then $\overline{g} \leq 3\omega$.
When the signature of the type is $\mathrm{RH}_{(-)}$, then

$$2e\overline{g} - (e-3)D^2 = 2e\overline{g} + (e-3)(\omega - 3\overline{g})$$
$$= (e-3)\omega - (e-9)\overline{g}$$

and hence,

$$(e-3)\omega - (e-9)\overline{g} = \tilde{\theta_3} + (p+u)Y + 3\tilde{\mathcal{Z}}.$$

Thus,

$$(e-3)\omega \geq (e-9)\overline{g}.$$

If $e \geq 12, \nu_1 \geq 3$ then $\overline{g} \leq 3\omega$.

20. ESTIMATE

We shall give some estimates of g in terms of ω.
Assuming that $\nu_1 \geq 3$ and $\sigma \geq 8$, we get $3\omega - g \geq 0$.
Thus,denoting $3\omega - g$ by F_4. Then $F_4 \geq 0$.

In that follows, we shall enumerate all types of pairs (S, D) satisfying that $F_4 \leq 9$ in case the sign of the type (S, D) is $\mathrm{RH}_{(+)}$.

When the sign of the type is $\mathrm{RH}_{(-)}$, then Assume that $\nu_1 \geq 3$ and $e \geq 12$. We shall enumerate all types of pairs (S, D) satisfying that $F_4 \leq 9$.

First, we consider in the case of $\mathrm{RH}_{(+)}$.

20.1. **case in which $u > 0$.** Replacing $3\omega - g$ by F_4, we get

$$(\sigma - 6)F_4 - 2(\sigma - 8)\omega = \tilde{\Theta}_2. \tag{6}$$

20.2. **$B \geq 3$.** If $B \geq 3$, then $\widetilde{B} - 2\sigma \geq \sigma$ and so

$$\tilde{\theta}_2 = \sigma\left(\widetilde{B} - 2\sigma\right) \geq \sigma^2. \tag{7}$$

Hence,

$$(\sigma - 6)F_4 - 2(\sigma - 8)\omega = \tilde{\Theta}_2 \geq \tilde{\theta}_2 \geq \sigma^2.$$

From $\sigma^2 + 4\sigma - 32 = (\sigma - 6)(\sigma + 10) + 28$, it follows that

$$9 \geq F_4 \geq 10 + \frac{28}{\sigma - 6} \geq 19.$$

Therefore, in that follows we assume $B \leq 2$.

20.3. $\widetilde{\mathcal{Z}} > 0$. Suppose that $\widetilde{\mathcal{Z}} > 0$. Then $\widetilde{\mathcal{Z}} \geq \nu_1 - 1$ and $Y \geq \nu_1 + \nu_1 - 1 = 2\nu_1 - 1$.

Hence,

$$(\sigma - 6)F_4 - (\sigma - 8)\omega = \tilde{\Theta}_2 \geq 2\sigma + p(2\nu_1 - 1) + 2(\nu_1 - 1).$$

By hypothesis,

$$\begin{aligned} p(2\nu_1 - 1) + 2(\nu_1 - 1) &= 2(p+2)\nu_1 - p - 2 \\ &= (p+1)(\sigma - p) - p - 2. \end{aligned}$$

Recalling that $F_4 \leq 9$ and assuming $\omega \geq 4$, we get

$$\begin{aligned} 9(\sigma - 6) &\geq 2(\sigma - 8)\omega + \tilde{\theta_2} + (p+1)(\sigma - p) - p - 2 \\ &\geq 8(\sigma - 8) + 2(\sigma - 8)\omega + \tilde{\theta_2} + (p+1)(\sigma - p) - p - 2. \end{aligned}$$

Hence, by $\sigma = p + 2\nu_1 \geq p + 6$, we get

$$10 + p^2 + 2p + 2 \geq \tilde{\theta_2} + p\sigma \geq \tilde{\theta_2} + p(p+6).$$

Therefore,

$$12 \geq \tilde{\theta_2} + 4p \geq 4p.$$

This implies that $p \leq 3$.

Hence, if $B = 0, 2$ and $u \geq 1$, then

$$12 \geq \tilde{\theta_2} + 4p \geq 2u\sigma + 4p \geq 2(p+6) + 4p = 6p + 12$$

Thus if $u > 0$ then $p = 0$. Otherwise, $p \leq 3$.

20.4. $\mathbf{RH}_{(+)}$. If $B = 1$ and $\mathrm{RH}_{(+)}$ then $\tilde{\theta_2} = (2u - p)\sigma \geq 0$. By the similar argument, if $2u - p = 0$ then $2u = p \leq 3$. Hence, $u = 1, p = 2$.

$$9(\sigma - 6) \geq 2(\sigma - 8)\omega + 2\sigma + \sigma - 2.$$

Replacing σ by $2\nu_1$ we obtain

$$3\nu_1 - 13 \geq (\nu_1 - 4)\omega.$$

Hence,

$$\frac{3\nu_1 - 13}{\nu_1 - 4} = 3 - \frac{2}{\nu_1 - 4} \geq \omega.$$

This induces that $\omega < 3$. But we supposed that $\omega > 4$.

20.5. **$\mathbf{RH}_{(-)}$.** Suppose that $B = 1$ and $3\sigma - 2e > 0$, in other words, the type is $\mathrm{RH}_{(-)}$.

By second Hartshorne's identity,

$$(e-3)\omega - (e-9)\overline{g} = \tilde{\theta_3} + (p+u)Y + 3\tilde{\mathcal{Z}}.$$

Hence, by making use of $F_4 = 3\omega - \overline{g}$,

$$(e-9)F_4 = 2(e-12)\omega + \tilde{\Theta_3}. \tag{8}$$

We suppose that $F_4 \leq 9$, for simplicity.

Then

$$\tilde{\Theta_3} = (e-9)F_4 - 2(e-12)\omega \leq 9(e-9) - 2(e-12)\cdot 4 = e+15. \tag{9}$$

By hypothesis saying that $\widetilde{\mathcal{Z}} > 0$, we obtain

$$\widetilde{\mathcal{Z}} \geq \nu_1 - 1, Y \geq 2\nu_1 - 1$$

Hence,

$$\tilde{\Theta_3} = \tilde{\theta_3} + (p+u)Y + \widetilde{\mathcal{Z}}\tilde{\theta_3} + \geq (2\nu_1 - 1) + 6\nu_1 - 6. \tag{10}$$

Thus,by $\tilde{p}$ denoting $p+u$, we obtain

$$\tilde{\Theta}_3 = \tilde{\theta}_3 + (p+u)Y + \widetilde{\mathcal{Z}}\tilde{\theta}_3 + \geq (2\nu_1 - 1) + 6\nu_1 - 6.$$

$$\begin{aligned}
\tilde{\Theta}_3 &\geq \tilde{\theta}_3 + \tilde{p}(2\nu_1 - 1) + 3\nu_1 - 3 \\
&= \tilde{\theta}_3 + (2\tilde{p} + 3)\nu_1 + \tilde{p} - 3 \\
&= \tilde{\theta}_3 + (2\tilde{p} + 3)(\frac{e - \tilde{p}}{3}) + \tilde{p} - 3 \\
&= \tilde{\theta}_3 + \frac{e(2\tilde{p} + 3)}{3} - \frac{2\tilde{p}^2 + 6\tilde{p} + 9}{3} \\
&\geq \tilde{\theta}_3 + e + \frac{2\tilde{p}e}{3} - \frac{2\tilde{p}^2 + 6\tilde{p} + 9}{3}.
\end{aligned}$$

By the inequality (9), we obtain

$$15 \geq \tilde{\theta}_3 + \frac{2\tilde{p}e}{3} - \frac{2\tilde{p}^2 + 6\tilde{p}}{3} - 3. \tag{11}$$

Thus,

$$\begin{aligned} 18 + \frac{2\tilde{p}^2 + 6\tilde{p}}{3} &\geq \tilde{\theta}_3 + \frac{2\tilde{p}e}{3} \\ &\geq \tilde{\theta}_3 + \frac{2\tilde{p}(\tilde{p} + 9)}{3}. \end{aligned}$$

Therefore,

$$18 \geq \tilde{\theta}_3 + 4\tilde{p}. \tag{12}$$

However,

$$\tilde{\theta}_3 = (p - 2u)(u + \nu_1) \geq \nu_1 \geq 3.$$

Hence,

$$15 = 18 - 3 \geq 4\tilde{p}. \tag{13}$$

We conclude that $\tilde{p} \leq 3$.

1. $\tilde{p} = 3$. Then from the inequality (**??**), it follows that

$$27 = 15 + 12 \geq 2e.$$

Hence,

$$\tilde{p} + 3\nu_1 e \leq 13.$$

This implies that $\nu_1 = 3$ and $\tilde{p} = p + u = 3$.

Since $p > 2u$ and $3 = p+u > 3u$, it follows that $p = 3, u = 0$. Moreover, $\sigma = p + 2\nu_1 = 9$ and $e = \tilde{p} + 3\nu_1 = 3 + 3 \cdot 3 = 12$.

The type turns out to be $[9 * 12, 1; 3^r]$.

$$\overline{g} = 51 - 3r, \omega = 18.$$

Hence, $F_4 = 3 + 3r$. Here, $r \leq 2$.

(1) $[9 * 12, 1; 3]$ has $F_4 = 6$.
(2) $[9 * 12, 1; 3^2]$ has $F_4 = 9$.

2. $\tilde{p} = 2$. Then $u = 0$ and $p = 2$. Hence, $e = 2 + 3 \cdot 3 = 11 < 12$. This contradicts the hypothesis.

3. $\tilde{p} = 1$. Then $u = 0$ and $p = 1$. Thus, $\nu_1 = 4$ and $e = 13$. Then $F_4 = 13$, which contradicts the hypothesis.

20.6. **equisingularity.** Suppose that $\widetilde{\mathcal{Z}} = 0$. Then the pair has equisingularity, in other words, $Y = r\nu_1$.

First suppose that $r \geq 6$. Then

$$\tilde{\Theta}_2 = 2u\sigma + pr\nu_1 \geq 2u\sigma + 6p\nu_1.$$

But $6p\nu_1 = 3p(\sigma - p)$ and so

$$9(\sigma - 6) \geq 10(\sigma - 8) + 2\sigma + 3p(\sigma - p).$$

Hence,

$$26 \geq \sigma + 3p(\sigma - p).$$

Thus,

$$26 + 3p^2 \geq (3p + 1)\sigma \geq (3p + 1)(p + 6).$$

We get $26 \geq 19p$. Therefore, $p = 1$.

Therefore,

$$9(\sigma - 6) \geq 2(\sigma - 8)\omega + 3\sigma - 3.$$

We get

$$6\sigma - 51 \geq 2(\sigma - 8)\omega.$$

Hence, $\sigma \geq 9$ and

$$\frac{6\sigma - 51}{\sigma - 8} \geq 2\omega.$$

$\sigma \geq 9$ implies that $\frac{6\sigma-51}{\sigma-8} < 6$. Hence, $\omega \leq 2$. A contradiction.

Suppose that $r \leq 5$. Then

$$\overline{g} = \sigma(\sigma - 2) + u(\sigma - 1) - r\frac{\nu_1(\nu_1 - 1)}{2},$$

$$\omega = \sigma(\sigma - 6) + u(\sigma - 3) - r\frac{\nu_1(\nu_1 - 3)}{2}.$$

Hence,

$$F_4 = 2\sigma(\sigma - 8) + 2u(\sigma - 4) - r\nu_1(\nu_1 - 4).$$

$2\sigma(\sigma - 8) \geq 8\nu_1(\nu_1 - 4)$ and

$$F_4 \geq 8\nu_1(\nu_1 - 4) - r\nu_1(\nu_1 - 4) = (8 - r)\nu_1(\nu_1 - 4).$$

If $\nu_1 > 4$ and $r \leq 5$ then $F_4 > 9$.

If $\nu_1 = 4$ and $r \leq 5$ then $\sigma = p + 8$ and $F_4 = 2\sigma(\sigma - 8) + 2u(\sigma - 4) = 2(p+8)p + 2u(p+4) > 9$.

Suppose that $\nu_1 = 3$ and $\sigma = p + 6 \geq 8$. Hence, $p \geq 2$ and so $F_4 = 2(6+p)(p-2) + 2u(p+2) + 3r$.

If $p \geq 3$ then

$$F_4 \geq 18 + 3r.$$

If $p = 2$ then

$$F_4 = 8u(p+2) + 3r.$$

By $F_4 \leq 9$ we get $u = 0$ and $F_4 = 3r$.

21. FORMULA OF F_4

If$B = 0, 2$ then

$$\begin{aligned} F_4 &= \sigma\tilde{B} - 4\tilde{B} - 8\sigma \\ &= 2\sigma(\sigma - 8) + 2u(\sigma - 4) \\ &= 8\nu_1(\nu_1 - 4) + 8(\nu_1 - 2)p + 2p^2 + 2u(p + 2(\nu_1 - 2)). \end{aligned}$$

If $\nu_1 = 4$ then $F_4 = 2p^2 + 16p + 2u(p + 4) \geq 0$. Hence, if $0 < F_4 < 10$ then $P = 0, u = 1$ and $F_4 = 8$. The type becomes $[8 * 9; 4^r]$, where $0 < r < 8$.

If $\nu_1 = 3$ then $F_4 = 2p^2 + 8p - 24 + 2u(p + 2) \geq 0$. Hence, if $F_4 = 0, \sigma \geq 8$ then $F_4 \geq 0$.

Moreover, if $p = 2, u > 0$ then $F_4 = 8u \geq 8$.

Thus $0 < F_4 < 10$ then $P = 2, u = 1$ and $F_4 = 8$. The type becomes $[8 * 9; 1]$.

If$B = 1, e \geq 12$ then

$$
\begin{aligned}
F_4 &= \sigma\tilde{B} - 4\tilde{B} - 8\sigma \\
&= 8\nu_1(\nu_1 - 1) + 2\nu_1(3p + 2u) + p^2 - 12p + 2u(p - 4).
\end{aligned}
$$

If $\nu_1 \geq 4$ then
$2\nu_1(3p+2u)+p^2-12p+2u(p-4) \geq 8(3p+2u)+p^2-12p+2u(p-4) \geq 8u$.
Thus $0 < F_4 < 10$ then $P = 0, u = 1$ and $F_4 = 8$. The type becomes $[8 * 13, 1; 4^r]$.

If $\nu_1 = 3, e = 6 + p + u \geq 12$ then $F_4 = -24 + p(p + 2u) + 6p + 4u \geq 0$.

21.1. **case in which $u = 0$ and $p \geq 1$.** Suppose that $u = 0$ and $p \geq 1$.

Moreover, suppose that $\widetilde{\mathcal{Z}} > 0$. Then $\widetilde{\mathcal{Z}} \geq \nu_1 - 1$ and $Y \geq \nu_1 + 2$.

Hence,

$$(\sigma - 6)F_4 - (\sigma - 8)\omega = \tilde{\Theta}_2 \geq p(\nu_1 + 2) + 2(\nu_1 - 1).$$

From $p(\nu_1 + 2) + 2(\nu_1 - 1) = \frac{(p+2)\sigma}{2} + \frac{-p^2+2p-4}{2}$ it follows that

$$\frac{p(p+2)}{2} + 26 \geq \sigma + \frac{(p+2)\sigma}{2}.$$

Hence, by $\sigma \geq p + 6$, we get

$$p^2 + 2p + 52 \geq (p + 4)\sigma \geq (p + 4)(p + 6) = p^2 + 10p + 24.$$

Hence, $28p \geq 8p$. This implies that $p \leq 3$.

$$\tilde{\Theta}_2 = 2u\sigma + pr\nu_1 \geq 2u\sigma + 6p\nu_1.$$

Finally, we consider in the case of $\mathrm{RH}_{(-)}$. Rewriting the identity, we get

$$\tilde{\Theta}_3 = (e-9)F_4 - 2(e-12)\omega.$$

From $9 \geq F_4$ and $\omega \geq 4, e \geq \tilde{p}+9$, it follows that

$$18 - Lu + \frac{(\tilde{p}-3+L)\tilde{p}}{3} \geq \frac{(\tilde{p}+3+L)(\tilde{p}+9)}{3}.$$

Further, we get

$$18 - Lu \geq \frac{(13-L)\tilde{p}+9}{3}.$$

Hence, $45 - 3L \geq (13-L)\tilde{p}$. This implies $\tilde{p} \leq 3$.

If $u > 0$ then $\tilde{p} = p + u > 3u \geq 3$. Therefore, $u = 0$.

We say that the sign of the type (S, D) is $\mathrm{RH}_{(+)}$ if either $B \neq 1$ or $B = 1$ and $2e - 3\sigma \geq 0$. Otherwise, we say that the sign of the type is $\mathrm{RH}_{(-)}$, namely in the case when $B = 1$ and

21.2. **case in which** $k = 0$**.** Suppose that $u = p = 0$ and $\widetilde{\mathcal{Z}} > 0$. Then

$$(\nu_1 - 3)F_4 \geq 2(\nu_1 - 4)\omega + \widetilde{\mathcal{Z}} \geq 8(\nu_1 - 4)\omega + \widetilde{\mathcal{Z}}.$$

By $F_4 \leq 9$, we obtain

$$\nu_1 + 5 \geq \widetilde{\mathcal{Z}}.$$

First, suppose that $\widetilde{\mathcal{Z}} = \nu_1 - 1$. Then the type turns out to be $[2\nu_1 * \sigma = 2\nu_1; \nu_1^{r-1}, \nu_1 - 1]$. Here, $\sigma = 2\nu_1$.

Then

- $\overline{g} = \frac{\nu_1(\nu_1 - 1)}{2}(8 - r) + \nu_1 - 1.$
- $\omega = \frac{\nu_1(\nu_1 - 3)}{2}(8 - r) + \nu_1 - 2.$
- $F_4 = \nu_1(\nu_1 - 4)(8 - r) + 2\nu_1 - 5.$

If $r = 8$ then $F_4 = 2\nu_1 - 5$. Assuming that $F_4 = 2\nu_1 - 5 \leq 9$, we get $\nu_1 \leq 7$.

TABLE 4. $[2\nu_1 * \sigma = 2\nu_1; \nu_1^{r-1}, \nu_1 - 1]$.

ν_1	$\overline{g}$	F_4	
4	3	2	3
5	4	3	5
6	5	4	7
7	6	5	9
8	7	6	11

Second, suppose that $\widetilde{\mathcal{Z}} > \nu_1 - 1$. Then

$$\nu_1 + 5 \geq \widetilde{\mathcal{Z}} \geq 2\nu_1 - 4.$$

Thus, $\nu_1 \leq 9$.

21.3. $\nu_1 = 9$. Suppose that $\nu_1 = 9$, and then

$$14 = \nu_1 + 5 \geq \widetilde{\mathcal{Z}} \geq 2\nu_1 - 4 = 14.$$

$$\widetilde{\mathcal{Z}} = 14, \widetilde{\mathcal{Z}} = 8x_1 + 14x_2 + 18x_3 + \cdots \ ,$$

Here, $x_1 = t_8, x_2 = t_7 + t_2, x_3 = t_6 + t_3, \cdots$.

Therefore,$t_7 + t_2 = 1$. $F_4 = 9, \omega = 4, \overline{g} = 3$.

However, if $t_7 = 1$, then

$$3 = \overline{g} = 18 \cdot 16 - 36(r-1) - 21.$$

Then $25 = 3r$, a contradiction.

However, if $t_2 = 1$, then

$$3 = \overline{g} = 18 \cdot 16 - 36(r-1) - 1.$$

Then $9(r-1) = 71$, a contradiction.

Hence, $\widetilde{\mathcal{Z}} > 2\nu_1 - 4$. In this case, $\widetilde{\mathcal{Z}} \geq 2\nu_1 - 2$. Thus $\nu_1 \leq 7$.

21.4. $\nu_1 = 7$. Then $\widetilde{\mathcal{Z}} = 12$

$$12 \geq 4F_4 - 6\omega = 12.$$

By $2F_4 - 3\omega = 6, \omega = 4, F_4 = 9$, we obtain $\overline{g} = 3$.

Note that

$$\widetilde{\mathcal{Z}} = 6x_1 + 10x_2 + 12x_3.$$

Hence, 1) $x_1 = 2$, 2) $x_3 = 1$.

1) $x_1 = 2$. The type becomes $[14 * 14; 7^{r-2}, 6^2]$.

$$3 = \overline{g} = 14 \cdot 12 - 21(r-2) - 30.$$

Then

$$33 = 14 \cdot 12 - 21(r-2).$$

A contradiction.

2) $x_3 = t_3 + t_4 = 1$. The type becomes $[14 * 14; 7^{r-2}, 3]$ or $[14 * 14; 7^{r-2}, 4]$.

21.5. $\nu_1 = 6$. Then $11 \geq \widetilde{\mathcal{Z}} = 10$

$$12 \geq 4F_4 - 6\omega = 12.$$

By $2F_4 - 3\omega = 6, \omega = 4, F_4 = 9$, we obtain $\overline{g} = 3$.

21.6. $\nu_1 = 5$. Then $F_4 = 40 + 4t_2 + 3t_3 - 5t_5$. If $F_4 < 10$ then $t_5 = 7$.

- $\overline{g} = 10 - (t_2 + 3t_3 + 6t_4)$.
- $\omega = 10 + t_2 - 2t_4$.
- $F_4 = 5 + 4t_2 + 3t_3$.

Since $\overline{g} = 10 - (t_2 + 3t_3 + 6t_4) \geq -1$, it follows that $t_4 = 1$, or 0.

By $F_4 = 5 + 4t_2 + 3t_3 < 10$, we see that $t_2 + t_3 \leq 1$.

21.7. **the case when $F_4 \leq 9$.**

Theorem 7. *Assume* $F_4 = 3\omega - \overline{g} \leq 9$. *Then*

(1) $\nu_1 = 3$.

(a) $[8 * 8; 3^{t_3}, 2^{t_2}]$, $F_4 = 3t_3 + 4t_2; 3t_3 + 4t_2 \leq 9$.

(b) $[8 * 9; 1]$, $F_4 = 8$.

(2) $\nu_1 = 4$.

(a) $[8 * 8; 4^{t_4}, 3^{t_3}, 2^{t_2}]$, $F_4 = 3t_3 + 4t_2; 3t_3 + 4t_2 \leq 9, 49 - (6t_4 + 4t_3 + t_2) \geq 0$.

(b) $[8 * 9; 4^{t_4}]$, $F_4 = 8, t_4 \leq 9$.

(3) $\nu_1 = 5$.

(a) $[10 * 10; 5^7, 4], F_4 = 5;$

(b) $[10 * 10; 5^7, 4, 3], F_4 = 8,$

(c) $[10 * 10; 5^7, 4, 2], F_4 = 9,$

(d) $[10 * 10; 5^7], F_4 = 5,$

(e) $[10 * 10; 5^7, 3], F_4 = 8,$

(f) $[10 * 10; 5^7, 2], F_4 = 9$

22.1. **invariant** $\boldsymbol{A}$**.** Since $A = Z^2 - \overline{g}$, it follows that

$$\begin{aligned}\sigma Z^2 - 2(\sigma - 2)\overline{g} &= (\overline{g} + A) - 2(\sigma - 2)\overline{g} \\ &= \sigma A - (\sigma - 4)\overline{g}.\end{aligned}$$

Hence,

$$\sigma A - (\sigma - 4)\overline{g} = \theta_2{}^* + p\overline{Y} + 2\mathcal{Z}^*.$$

When the signature of the type is $\mathrm{RH}_{(+)}$, then

$$\sigma A \geq (\sigma - 4)\overline{g}.$$

If $\sigma \geq 6$, then $\overline{g} \leq 3A$.

When the signature of the type is $\mathrm{RH}_{(-)}$, then

$$\begin{aligned}eZ^2 - 2(e - 3)\overline{g} &= e(A + \overline{g}) - 2(e - 3)\overline{g} \\ &= eA - (e - 6)\overline{g}.\end{aligned}$$

Moreover,

22.2. **invariants A and α.** We obtain the following identities:

$$(\sigma - 2)\alpha - 2(\sigma - 4)\overline{g} = \tilde{\Theta}_2. \tag{14}$$

$$(e - 3)\alpha - 2(e - 6)\overline{g} = \tilde{\Theta}_3. \tag{15}$$

$$(\sigma - 2)\omega - (\sigma - 6)\overline{g} = \tilde{\Theta}_2. \tag{16}$$

$$(e - 3)\omega - (e - 9)\overline{g} = \tilde{\Theta}_3. \tag{17}$$

$$\sigma A - (\sigma - 4)\overline{g} = {\Theta_2}^*. \tag{18}$$

$$eA-(e-6)\overline{g}=\Theta_3{}^{*}. \tag{19}$$

By $\Theta_{2-1}{}^{*}$ we denote $2\Theta_2{}^{*}-\tilde{\Theta}_2$, then

$$2\sigma A-(\sigma-2)\alpha=\Theta_{2-1}{}^{*}.$$

When the sign of the type is $\mathrm{RH}_{(+)}$, then $\Theta_{2-1}{}^{*}\geq 0$ and so

$$2\sigma A\geq(\sigma-2)\alpha.$$

By $\Theta_{3-1}{}^{*}$ we denote $2\Theta_3{}^{*}-\tilde{\Theta}_3$, that is non-negative and so

$$2eA-(e-3)\alpha=\Theta_{3-1}{}^{*}.$$

When the sign of the type is $\mathrm{RH}_{(-)}$, then $\Theta_{3-1}{}^{*} > 0$ and so

$$2eA \geq (e-3)\alpha.$$

22.3. **estimates.**

Proposition 6. *Suppose that either* $\sigma \geq 8, \nu_1 \geq 3$ *or* $e \geq 12, \nu_1 \geq 3$.

(1) *If* ω *is even,* $\overline{g} \leq 3\omega$. *The identity occurs whenever* $[8 * 8; 4^r]^*, 1 \leq r \leq 7$.
(2) *If* $\overline{g} < 3\omega$ *then* $\overline{g} \leq 3\omega - 3$. *The identity occurs whenever* $[8 * 8; 4^r, 3]^*, 0 \leq r \leq 7$.
(3) *If* $\overline{g} < 3\omega - 3$ *then* $\overline{g} \leq 3\omega - 5$.
(4) *If* ω *is odd, then* $\overline{g} \leq 3\omega - 4$. *The identity occurs whenever* $[8 * 8; 4^r, 2]^*, 1 \leq r \leq 7$.

22.4. **case when** $k = 0$**.** Suppose that $p = u = 0$. Then $(\sigma - 6)F_4 - 2(\sigma - 8)\omega = \tilde{\Theta}_2$ turns out to be

$$(\nu_1 - 3)F_4 - 2(\nu_1 - 4)\omega = \widetilde{\mathcal{Z}}$$

First, when the type is $[2\nu_1 * 2\nu_1; \nu_1^{r-1}, \nu_1 - 1]$ we compute ω, F_4.

(1) $\overline{g} = \frac{\nu_1(\nu_1-1)}{2}(8 - r) + \nu_1 - 1,$

(2) $\omega = \frac{\nu_1(\nu_1-3)}{2}(8 - r) + \nu_1 - 2,$

(3) $F_4 = \nu_1(\nu_1 - 4)(8 - r) + 2\nu_1 - 5,$

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