SUPER PERFECT NUMBERS AND SUPER TWIN PRIMES

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1. Super twin primes

Given integers $(a > 0, b)$, if q and $p = aq + b$ are both primes (> 2) then $(p, q)$is said to be super twin primes with respect to a, b. (ST Primes.)

Moreover, given integers $(a > 0, b, c > 0, d)$, if the numbers $p = aq + b, r = cq + d$ and r are all primes then (p, q, r) are said to be ultra triplet primes with respect to a, b, c, d.

Iitaka proposed the next problem:

Suppose that there exit infinitely many super twin primes. What requirements are needed for a, b ?

Takahashi replied that the following condition may be sufficient.

(i)$a + b \equiv 1 \mod 2$, (ii)a, b are relatively prime.

Later Takahashi made an approximation formula of numbers of supertwin primes or ultra triplet primes by making use of integrals of functions $\dfrac{1}{\log(t)\log(at+b)}$,which is similar to formulas by Hardy and Littlewood.

Table 1. Numbers of super twin primes $p, q = 3p + 10$, results by approximate formula of Takahashi

$p \leq x$	formula of T.	N. of ST Primes E	$D = T - E$	$D/E * 10000$
1,000	96	79	17	2,091
10,000	492	472	20	430
100,000	2,993	2,941	52	176
500,000	11,275	11,183	92	81
1,000,000	20,201	20,210	9	5
2,000,000	36,411	36,359	52	14
5,000,000	79,984	79,869	115	14
10,000,000	145,850	145,758	92	6

2. Perfect numbers with translation parameter m

By $\sigma(a)$ we denote the sum of divisors of a positive integer a.

If $\sigma(a) = 2a$ then a is said to be perfect number. For example,$a = 6, 28, 496, 8128, 33550336,$

Euclid showed that if $q = 2^{e+1} - 1$ is prime, then $a = 2^e q$ turns out to be a perfect number.

Given m ,assume that $q = 2^{e+1} - 1 + m$ is prime and $\alpha = 2^e q$. Then such α is said to be perfect number with translation parameter m.

Putting $a = 2^e$ and $N = 2^{e+1} - 1$, we see that $N = \sigma(a) = 2a - 1, q = N + m = \sigma(a) + m, q + 1 = 2a + m.$

Making use of $Nq = (2a - 1)q = 2\alpha - q, N - q = -m$, we obtain

$$\begin{aligned} \sigma(\alpha) &= \sigma(2^e)\sigma(q) \\ &= Nq + N \\ &= 2\alpha - q + N \\ &= 2\alpha - m. \end{aligned}$$

In general, natural number α satisfying $\sigma(\alpha) = 2\alpha - m$, is said to be perfect number (perfect number with translation parameter m in the general sense.

If $m = 0$ and α is even , such perfect number is written $\alpha = 2^e q$ where ($q = 2^{e+1} - 1$ is prime).

3. Super perfect numbers

We use the symbol: $\sigma^2(a) = \sigma(\sigma(a))$.

If $\sigma^2(a) = 2a$ is satisfied then a is said to be super perfect number,by D.Suryanaryana, in 1969.

Theorem 1 (D.Suryanaryana)**.** *Even a with $\sigma^2(a) = 2a$ turn out to 2^e parts of perfect numbers.*

The result looks like Euler' theorem saying that even perfect numbers turn out to be $2^e q, q = 2^{e+1} - 1$:prime.

Table 2. $\sigma^2(a) = 2a$ (super perfect number)

a	factor	q	q factor
2	2	3	3
4	2^2	7	7
16	2^4	31	31
64	2^6	127	127
4096	2^{12}	8191	8191
65536	2^{16}	131071	131071

4. Super perfect numbers with translation m ,base P

Let a prime P be a base .

Super perfect numbers with translation m are defined as follows.

Suppose $a = P^e$ and translation parameter m. Assume $q = \sigma(a) + m$ is prime.

Hence $\sigma(q) = q + 1$.

Putting $A = \sigma(a) + m$, we get $\sigma(A) = q + 1$.

Letting $\overline{P} = P - 1$ and $W = P^{e+1} - 1$, we obtain

$q - m = \sigma(a) = \dfrac{W}{\overline{P}}$,

$$\sigma(A) = q+1 = \frac{W}{\overline{P}}+1+m = \frac{W + \overline{P}(1 + m)}{\overline{P}} = \frac{P^{e+1} + P - 2 + m\overline{P}}{\overline{P}}$$

.

Thus

$$\overline{P}\sigma(A) = aP + P - 2 + m\overline{P}.$$

Definition 1. $A = \sigma(a) + m, \overline{P}\sigma(A) = aP + P - 2 + m\overline{P}$ *is said to be simultaneous defining equations of super perfect numbers with translation parameter* m *and base* P.

Definition 2. *If* a *satisfies*

$A = \sigma(a) + m, \overline{P}\sigma(A) = aP + P - 2 + m\overline{P}$

then it is said to be super perfect numbers with translation parameter m *and base* P.

A is said to be partner of super perfect number a.

Lemma 1. *If* $a = P^e$*, then partner* A *becomes prime.*

Proof.

Easy to check that $\sigma(A) = A + 1$.

5. When $P = 3$

Examples.

If $P = 3$ then

$$A = \sigma(a) + m, 2\sigma(A) = 3a + 1 + 2m.$$

Table 3. $P = 3, m = -2$: super perfect number

a	factor	A	factor
3	3	2	2
9	3^2	11	11
49	7^2	55	5*11
729	3^6	1091	1091
6561	3^8	9839	9839

TABLE 4. $P = 3, m = 0$: super perfect numbers

a	factor	A	factor
9	3^2	13	13
729	3^6	1093	1093
531441	3^{12}	797161	797161

6. Analog of Euler's theorem

Theorem 2. *If a satisfies $A = \sigma(a), \overline{P}\sigma(A) = aP + P - 2$ (i.e., $m = 0$) and moreover, if a is a multiple of P , then $a = P^e$ for some $e > 0$.*

The result is a generalization of Suryanaryna's result, which looks like Euler's theorem.

Proof.

Since a is a multiple of P, we obtain $a = P^e L, (P \not| L)$.

By $W = P^{e+1} - 1$, it follows that $A = \sigma(a) = \dfrac{W\sigma(L)}{\overline{P}}$.

Letting $W_0 = 1 + P + P^2 + \cdots + P^e$,we get $W_0 = \dfrac{W}{\overline{P}}$.

By $A = \sigma(a) = W_0\sigma(L)$, from $L > 1$, it follows that ,$\sigma(L) \geq 1 + L$.

$$\sigma(A) \geq 1 + A + \sigma(L) > 1 + W_0\sigma(L) + L.$$

By multiplying $\overline{P}$,

$\overline{P}\sigma(A) > \overline{P} + \overline{P}W_0\sigma(L) + \overline{P}L = \overline{P} + W\sigma(L) + \overline{P}L.$

Since $W = P^{e+1} - 1$, we have

$$\begin{aligned} aP + P - 2 &= P^{e+1}L + P - 2 \\ &= (W+1)L + P - 2 \end{aligned}$$

Thus

$$\begin{aligned} aP + P - 2 &= \overline{P}\sigma(A) \\ &> \overline{P} + \overline{P}W_0\sigma(L) + \overline{P}L \\ &= \overline{P} + W\sigma(L) + \overline{P}L. \end{aligned}$$

Hence,

$(W+1)L+P-2 \geq \overline{P}+W\sigma(L) > P+1+WL+W+\overline{P}L.$

ocntradiction. Therefore, $L=1$ and then $a=P^e$.

End

A is said to be generalized Mersenne prime.

Table 5. $P = 2$: Mersenne primes

e	$q = 1 + P + \cdots + P^e$	$\alpha = aq$: perfect number
1	3	6
2	7	28
4	31	496
6	127	8128
12	8191	33550336
16	131071	
18	524287	
30	2147483647	
60	2305843009213693951	
88	618970019642690137449562111	

Table 6. $P = 3$: generalized Mersenne prime

e	$q = 1 + P + \cdots + P^e$
2	13
6	1093
12	797161
70	3754733257489862401973357979128773

Table 7. $P = 5, 7$: generalized Mersenne prime

$P = 5$	
e	$q = 1 + P + \cdots + P^e$
2	31
6	19531
10	12207031
12	305175781
$P = 7$	
4	2801
12	16148168401

7. $P = 3, m = -8$: SUPER PERFECT NUMBERS

In the case of $P = 3, m = -8$, there exist many prime solutions.

Defining equations are $A = \sigma(a) - 8, 2\sigma(A) = 3(a - 5)$.

Table 8. $P = 3, m = -8$: super perfect numbers

a	factor	A	factor	B	factor
9	3^2	5	5	5	5
81	3^4	113	113	113	113
$a = p$	factor	$A = 2Q$	factor	$B = \sigma(a) + 1$	factor
13	13	6	$2 * 3$	11	11
17	17	10	$2 * 5$	17	17
29	29	22	$2 * 11$	35	$5 * 7$
41	41	34	$2 * 17$	53	53
53	53	46	$2 * 23$	71	71
89	89	82	$2 * 41$	125	5^3
101	101	94	$2 * 47$	143	$11 * 13$
113	113	106	$2 * 53$	161	$7 * 23$
149	149	142	$2 * 71$	215	$5 * 43$
173	173	166	$2 * 83$	251	251
233	233	226	$2 * 113$	341	$11 * 31$
269	269	262	$2 * 131$	395	$5 * 79$
281	281	274	$2 * 137$	413	$7 * 59$
353	353	346	$2 * 173$	521	521
389	389	382	$2 * 191$	575	$5^2 * 23$

Table 9. $Q, p = 2Q + 7$: supertwin primes

Q	p	B	factor
2	11	8	2^3
3	13	11	11
5	17	17	17
11	29	35	$5 * 7$
17	41	53	53
23	53	71	71
41	89	125	5^3
53	113	161	$7 * 23$
71	149	215	$5 * 43$
83	173	251	251
113	233	341	$11 * 31$
131	269	395	$5 * 79$
137	281	413	$7 * 59$
173	353	521	521
191	389	575	$5^2 * 23$
197	401	593	593

Here $B = \sigma(A) - 1 = 3Q + 2$. If B is prime then (Q, p, B) are ultra triplet primes.

Assme we have prime solutions; $a = p$(:prime) .

Since $p - 7$ is even, suppose that $A = \sigma(a) + m = \sigma(p) - 8 = p - 7 = 2^{\varepsilon}Q, (Q :$ odd).

We shall show that $\varepsilon = 1$ and hence, $p - 7 = 2Q, Q$: prime.

Proof

Recalling $2\sigma(A) = 3a + 1 + 2m = 3a - 16 = 3p - 15$,we get $2\sigma(A) = 2N\sigma(Q)$ where $N = 2^{\varepsilon+1} - 1$.

Thus

$$2\sigma(A) = 2N\sigma(Q) = 3p - 15 = 3(p - 5).$$

By $p = 2^{\varepsilon}Q + 7$

$$2N\sigma(Q) = 3(p-5) = 3(p-7)+6 = 3*2^{\varepsilon}Q+6 = 6*(2^{\varepsilon-1}Q+1).$$

Putting $N_1 = 2^{\varepsilon-1}$, we have,$N = 4N_1 - 1$ and

$$8N_1 - 2\sigma(Q) = 6 * (N_1 Q + 1).$$

Hence,

$$3 * (N_1 Q + 1) = (4N_1 - 1)\sigma(Q).$$

$$\begin{aligned} 3 * (N_1 Q + 1) &= (4N_1 - 1)\sigma(Q) \\ &\geq (4N_1 - 1)(Q + 1) \\ &= (4N_1 - 1)Q + 4N_1 - 1 \\ &= 4N_1 Q - Q + 4N_1 - 1. \end{aligned}$$

Accordingly,

$$3 * N_1Q + 3 \geq 4N_1Q - Q + 4N_1 - 1.$$

Thus $4 \geq N_1Q + 4N_1 - Q$.

$$0 \geq (Q + 4)(N_1 - 1).$$

Hence, $N_1 = 1, \varepsilon = 1$.

By $3 * (N_1Q + 1) = (4N_1 - 1)\sigma(Q)$, we get

$3 * (Q + 1) = (4 - 1)\sigma(Q)$. , Since Q is prime ,$p = 7 + 2Q$ is also prime.Thus, $(Q, p = 2Q + 7)$ are supertwin primes.

Theorem 3. *Assume that* $P = 3, m = -8$.*If a super perfect number is prime* p, *then the partner* A *becoms* $2Q$ *(*Q*:prime),* $(Q, p = 2Q + 7)$ *are supertwin primes.*