On logarithmic plurigenera of algebraic plane curves (the fourth version)

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1 Introduction

Let C be an algebraic curve on the projective plane $\mathbf{P}^2$. Let $P_m[C]$ denote logarithmic m genus of pairs $(\mathbf{P}^2, C)$. $P_1[C]$ turns out to be the genus g of C. These are invariant under Cremona transformations,i.e., birational transformations between the projective plane $\mathbf{P}^2$ and itself.

To compute logarithmic plurigenera $P_m[C]$, one has to construct non-singular minimal pairs (S, D) which are birationally equivalent to the given $(\mathbf{P}^2, C)$. Let Z be $D + K_S$. Then Z^2, D^2, g where g is the genus of the curve D, are birational invariants as pairs. Moreover, $P_{2,1}[D] = \dim |2K_S + D| + 1$ is called, the (2,1) genus of a nonsingular pair (S, D) Inequalities among these invariants will be established.

The invariant bigenus $P_2[C]$ is very powerful to characterize algebraic plane curves of certain type , that was first recognized by Coolidge [2]. Actually he showed the next two results in 1928:

1) if $P_2[C] = 0$ then by a Cremona transformation, C is transformed into a line.

2) If $P_2[C] = 1$ then by a Cremona transformation, C is transformed into either a nonsingular cubic or a rational curve of degree $3m$ with nine m ple points and a double point.

The purpose of this paper is to extend his results. Actually, structure of plane curves C with $P_2[C] = 2, 3, Z^2 = 0, 1, 2, 3$ or $P_{2,1}[C] = 1, 2, 3$ will be determined.

2 minimal models of pairs

2.1 birational transformations between pairs

Here, we recall basic notions and results in birational geometry of pairs (see [5, Iitaka]). Let C be a curve on a non-singular projective surface S.

Two pairs (S, C) and (S_1, C_1) are said to be *birationally equivalent* , if there exists a birational map $f : S \to S_1$ such that the proper image $f[C]$ of C by f coincides with C_1. Here the proper image $f[C]$ is , by definition, the closure of the image $f(x)$ of the generic point x of C. When there is no danger of confusion, we say that C is birationally equivalent to C_1 as imbedded curves if two pairs (S, C) and (S_1, C_1) are birationally equivalent. f is said to be a birational transformation between pairs.

The purpose of birational geometry of pairs is to study properties of pairs (S, C) which are invariant under birational transformations.

A pair (W, D) is said to be non-singular , if both W and D are non-singular. In this case, we have complete linear systems $|m(D + K_W)|$ for any $m > 0$, where K_W indicates a canonical divisor on W. The dimension $\dim|m(D + K_W)| + 1$ depends on both D and W. But to simplify the notation, we use the symbol $P_m[D]$ to denote $\dim|m(D + K_W)| + 1$. Using this we define the *Kodaira dimension* $\kappa[D]$ of (W, D) to be the degree of $P_m[D]$ as a function in m. It is easy to see that $P_m[D]$ and $\kappa[D]$ are birational invariants of (W, D) in the above sense. Hereafter, we shall consider pairs (S, D) in which S is rational. Then $P_1[D]$ turns out to be the genus of D, denoted by $g(D)$.

A non-singular pair (S, D) is said to be *relatively minimal* , whenever the intersection number $D \cdot E \geq 2$ for any exceptional curve (of the first kind) E on S such that $E \neq D$. In this case every birational morphism from (S, D) into another non-singular pair (S_1, D_1) turns out to be isomorphic. Moreover, the pair (S, D) is said to be *minimal* , if every birational map from any non-singular pair (S_1, D_1) into (S, D) turns out to be regular. Any relatively minimal pair (S, D) is minimal if $\kappa[D] = 2$ (see Theorem I in [6]). In this case, the self-intersection number D^2 is a birational invariant. Moreover, if $\kappa[D] \geq 0$, D^2 is also a birational invariant except for the case in which $\kappa[D] = 0$ and $P_1[D] = 1$.

It is well known that given a rational surface S, after contracting all exceptional curves on S successively, we get relatively minimal models of S. Relatively minimal models of rational surfaces are the projective plane $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$ or a $\mathbf{P}^1-$ bundle over $\mathbf{P}^1$ which has a section Δ_∞ with negative self intersection number. The last surface is denoted by a symbol Σ_B where

$-B$ denotes the self intersection number ${\Delta_\infty}^2$ of the section Δ_∞. Here, we call Σ_B Hirzebruch surface of degree B after Kodaira.

For simplicity, we let Σ_0 denote the product surface $\mathbf{P}^1 \times \mathbf{P}^1$. The Picard group of $\Sigma_B (B \geq 0)$ is generated by the section Δ_∞ and a fiber $F_c = \rho^{-1}(c)$ of the $\mathbf{P}^1-$ bundle, where $\rho : \Sigma_B \to \mathbf{P}^1$ is the projection.

2.2 curves on Σ_B

Let C be an irreducible curve on Σ_B. Then there exist integers σ and e such that

$$C \sim \sigma \Delta_\infty + eF_c.$$

We have $C \cdot F_c = \sigma$ and $C \cdot \Delta_\infty = e - B \cdot \sigma$. Hereafter, suppose that $C \neq \Delta_\infty$. Thus $C \cdot \Delta_\infty \geq B$ and hence, $e \geq \sigma \cdot B$. If $B > 0$ then ${\Delta_\infty}^2 = -B < 0$ and such a section Δ_∞ is uniquely determined. For a surface $\Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$, we get $F_c \sim \mathbf{P}^1 \times$ point and $\Delta_\infty \sim$ point $\times \mathbf{P}^1$. We may assume that $e \geq \sigma$. Thus σ and e are uniquely determined for a given curve C on Σ_B.

Letting g_0 be the virtual genus of C and K_0 a canonical divisor on Σ_B, we get

$$\begin{aligned} 2g_0 - 2 &= C^2 + K_0 \cdot C \\ &= (\sigma \Delta_\infty + eF_c) \cdot ((\sigma - 2)\Delta_\infty + (e - B - 2)F_c) \\ &= B(1 - \sigma)\sigma + 2(e\sigma - e - \sigma). \end{aligned}$$

Hence,

$$g_0 = (e - 1)(\sigma - 1) - \frac{B\sigma(\sigma - 1)}{2},$$

$$C^2 = 2e\sigma - \sigma^2 b.$$

Letting $f = e - B\sigma = C \cdot \Delta_0 \geq 0$, we obtain

$$C \sim \sigma \Delta_0 + fF_c,$$

$$K_0 \sim -2\Delta_0 - (2 - B)F_c,$$

$$Z_0 = C + K_0 \sim (\sigma - 2)\Delta_0 + (f - 2 + B)F_c,$$

where Δ_0 is an irreducible curve linearly equivalent to $\Delta_\infty + BF_c$. Thus,

$$g_0 = (f - 1)(\sigma - 1) + \frac{B\sigma(\sigma - 1)}{2} = \frac{(\sigma - 1)(B\sigma + 2f - 2)}{2},$$

$$C^2 = 2f\sigma + \sigma^2 B = \sigma(2f + B\sigma),$$
$$C^2 = \frac{2\sigma}{\sigma - 1} g_0 + 2\sigma,$$
$$Z_0^2 = \frac{2\sigma - 4}{\sigma - 1} g_0 + 4 - 2\sigma.$$

2.3 types of pairs

We assume C to be singular. Let $\nu_1(C)$ denote the highest multiplicity of the singular point of C. We take a singular point p_1 on C with $\mathrm{mult}_{p_1}(C) = \nu_1(C)$, that is denoted by ν_1. Blowing up at center p_1, we obtain a surface S_1 and a proper birational morphism $\mu_1 : S_1 \to S_0 = \Sigma_B$, which satisfies

$$\mu_1^*(C) \sim C_1 + \nu_1 E_1,$$

where $E_1 = \mu_1^{-1}(p_1)$ and C_1 is the proper transform of C by μ_1^{-1}. Letting K_0 and K_1 denote canonical divisors of $S_0 = \Sigma_B$ and S_1, respectively, we have

$$K_1 \sim \mu_1^*(K_0) + E_1.$$

In order to simplify the notation, the total inverse images of divisors are denoted by the same symbol. For example, the above relation is denoted by

$$K_1 \sim K_0 + E_1.$$

Letting ν_2 denote $\nu_1(C_1)$ and taking p_2 on C_1 such that $\mathrm{mult}_{p_2}(C_1) = \nu_2$, we get a surface S_2 and a birational morphism $\mu_2 : S_2 \to S_1$ which is obtained by blowing up at p_2. Continuing this process, we obtain a sequence of birational morphisms $\mu_1, \mu_2, \cdots, \mu_r$ such that the composition μ of these morphisms gives rise to a minimal resolution of the singularities of the imbedded curve C:

$$W = S_r \xrightarrow{\mu_r} S_{r-1} \xrightarrow{\mu_{r-1}} \cdots \xrightarrow{\mu_2} S_1 \xrightarrow{\mu_1} S_0 = \Sigma_B.$$

Thus letting $\nu_j = \mathrm{mult}_{p_j}(C_{j-1})$, we get a sequence of integers $\nu_1, \nu_2, \cdots, \nu_r$ such that $\nu_1 \geq \nu_2, \cdots, \nu_r \geq 2$, where C_0 stands for C.

Definition 1 *The type of the pair* (Σ_B, C) *is defined to be* $[\sigma * e, B\,; \nu_1, \nu_2, \cdots, \nu_r]$ *and the type of a curve* C *on* $\mathbf{P}^2$ *is denoted by* $[d; \nu_0, \nu_1, \nu_2, \cdots, \nu_r]$ *where* d *is the degree of* C *and* $\nu_0, \nu_1, \nu_2, \cdots, \nu_r$ *denote the multiplicities of singular points of* C.

Occasionally, the curve C of a pair (Σ_B, C) is said to be a *curve of type* $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$. For simplicity, $[\sigma * e,\ 0\ ; \nu_1, \nu_2, \cdots, \nu_r]$ is rewritten as $[\sigma * e; \nu_1, \nu_2, \cdots, \nu_r]$.

In the case where C is itself non-singular, we put $r = 0$ or $r = 1$ and ν_1 $=1$ by convention.

3 Elementary transformations

3.1 $\mathbf{I}_+(p, \nu_1), \mathbf{I}_-(p, \nu_1)$

We shall introduce special kinds of birational transformations among Hirzebruch surfaces, called elementary transformations. Take a point p on Σ_B. Blowing up at p, we get a birational morphism $\mu : S_1 \to S_0 = \Sigma_B$. Then letting F be a fiber $\rho^{-1}(\rho(p))$ of Σ_B and letting E be the exceptional curve $\mu^{-1}(p)$, we obtain

$$\mu^*(\sigma\Delta_\infty + eF_c) \sim \mu^*(C) = C' + \nu_1 E,$$

$$\mu^*(F_c) \sim \mu^*(F) = F' + E.$$

Here $F_c = \rho^{-1}(c)$; F' and C' denote the proper inverse images of F and C, respectively, and ν_1 indicates the multiplicity of C at p.

If $p \in \Delta_\infty$, then denoting by Δ'_∞ the image of Δ_∞, we get $(\Delta'_\infty)^2 = -B - 1$. Moreover, $\mu^*(\Delta_\infty) = \Delta'_\infty + E$, and

$$C' \sim \sigma(\Delta'_\infty + E) + e(F' + E) - \nu_1 E.$$

Since $F'^2 = -1$, F' becomes an exceptional curve. Contracting F' into a non-singular point p' , we get a non-singular surface S' and a proper birational morphism $\mu' : S_1 \to S'$. By $\Delta'_\infty \cdot F' = \Delta_\infty \cdot F - 1 = 1 - 1 = 0, \mu'$ is isomorphic around Δ'_∞. Thus, the image Δ''_∞ of Δ'_∞ by μ' is isomorphic to Δ'_∞. Hence,

$$(\Delta''_\infty)^2 = {\Delta'_\infty}^2 = {\Delta_\infty}^2 - 1 = -B - 1.$$

This implies that S' is isomorphic to Σ_{B+1}. The image of C' by μ' is denoted by C_0, that satisfies

$$C_0 \sim \sigma'\Delta''_\infty + e'F_v,$$

for some integers σ' and e', where F_v is a fiber of the $\mathbf{P}^1-$ bundle Σ_{B+1}. The proper inverse image F' of F_v by μ' satisfies

$$\mu'^*(F_v) = F' + E.$$

Let ν_1' denote the multiplicity of C_0 at p'. Then

$$C' \sim \sigma'\Delta_\infty'' + e'(F' + E) - \nu_1' F'.$$

Since E, F' and Δ_∞'' are linearly independent, it follows that

$$\sigma' = \sigma, \quad \sigma + e - \nu_1 = e', \quad e = e' - \nu_1'.$$

Hence,

$$\nu_1' = \sigma - \nu_1, \quad e' = e + \nu_1' = e + \sigma - \nu_1.$$

Also in the case when $p \notin \Delta_\infty$, we get the similar result. Thus, the next proposition is established.

Proposition 1 1. *If* $p \in \Delta_\infty$ *then* $\sigma' = \sigma$, $S' = \Sigma_{B+1}$ and $\nu_1' = \sigma - \nu_1, e' = e + \nu_1'$.

2. If $p \notin \Delta_\infty$ then $\sigma' = \sigma$,$B > 0$ and $S' = \Sigma_{B-1}, \nu_1' = \sigma - \nu_1, e' = e - \nu_1$.

Note that in the case when $B = 1, p \notin \Delta_\infty$, S' becomes Σ_0 and $e' < \sigma'$ may happen.

The birational map $\mu \cdot \mu'^{-1}$ is called *elementary transformation of type* I. More precisely, if $p \in \Delta_\infty$ then the birational map $\mu \cdot \mu'^{-1}$ is said to be *the elementary transformation* $\mathrm{I}_+(p, \nu_1)$. If $p \notin \Delta_\infty$ then the birational map $\mu \cdot \mu'^{-1}$ is said to be the *elementary transformation* $\mathrm{I}_-(p, \nu_1)$.

NOTE: During the performance of an elementary transformation, the singular point with multiplicity ν_1 disappears and a singular point with multiplicity ν_1' appears if $\nu_1' > 0$.

Let (S, D) be a pair obtained from the pair (Σ_B, C) of type $[\sigma{*}e, B; \nu_1, \nu_2, \cdots, \nu_r]$ by minimal resolution of singularities of C. Moreover, let (S_0, D_0) be a pair obtained by minimal resolution of singularities from the pair (S, D) by the elementary transformation $\mathrm{I}_+(p, \nu_1)$ where $\nu_1 = \mathrm{mult}_p(C)$. Then if $\nu_1' \neq 1$, we get

$$D_0{}^2 - D^2 = C^2 - \nu_1{}^2 - C_0{}^2 + \nu_1'^2$$
$$= \sigma(2e - \sigma B) - \nu_1{}^2 - \sigma(2(e + \sigma - \nu_1) - \sigma(B + 1)) + \nu_1'^2 = 0.$$

Moreover, if $\nu_1' = 1$, then

$$D_0^2 - D^2 = 1.$$

Thus, D^2 increases.

In both cases, we write $D_0{}^2 - D^2 = \varepsilon(\mathrm{I}_-(p, \nu_1))$. Similarly, let (S, D) be a pair obtained by minimal resolution of singularities from the pair (Σ_B, C)

and let (S_0, D_0) be a pair obtained by minimal resolution of singularities from the pair (S, D) by the elementary transformation $\mathrm{I}_-(p, \nu_1)$. In this case, if $\nu_1' \neq 1$, then

$$D_0{}^2 - D^2 = 0$$

and moreover, if $\nu_1' = 1$, then

$$D_0{}^2 - D^2 = 1.$$

When $\sigma = 2\nu_1$ and $p_1 \in \Delta_\infty$, after performing an elementary transformation $\mathrm{I}_+(p, \nu_1)$ to a pair of type $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$, the new type denoted by $[\sigma * e', B'; \nu_1, \nu_2, \cdots, \nu_r]$ satisfies that $e' = e + \nu_1, B' = B + 1$ and then $g_0 = (e-1)(\sigma-1) - \dfrac{B\sigma(\sigma-1)}{2} = (e'-1)(\sigma-1) - \dfrac{B'\sigma(\sigma-1)}{2}$ and $C^2 = 2e\sigma + \sigma^2 B = 2e'\sigma + \sigma^2 B'$. Therefore, g_0 and C^2 are invariant under elementary transformations $\mathrm{I}_+(p, \nu_1)$ and $\mathrm{I}_-(p, \nu_1)$.

Therefore, starting from the type $[2\nu_1 * e; \nu_1, \nu_2, \cdots, \nu_r]$, we get the type $[2\nu_1 * (e+\nu_1), 1; \nu_1, \nu_2, \cdots, \nu_r]$ and $[2\nu_1 * (e+2\nu_1), 2; \nu_1, \nu_2, \cdots, \nu_r]$ provided that $e + 2\nu_1 \geq 4\nu_1$. Note that if $e \geq i\nu_1$ then the type $[2\nu_1 * (e+i\nu_1), i; \nu_1, \nu_2, \cdots, \nu_r]$ is possible.

Remark 1 *Moreover, if $e \geq i\nu_1$ then the types $[2\nu_1 * (e+i\nu_1), i; \nu_1, \nu_2, \cdots, \nu_r]$ for $1 \leq i \leq [e/\nu_1]$ are said to be the types associated with $[2\nu_1 * e, 0; \nu_1, \nu_2, \cdots, \nu_r]$.*

For example, the types associated with $[8 * 8; 4^6, 3^4]$ are $[8 * 12, 1; 4^6, 3^4]$ and $[8 * 16, 2; 4^6, 3^4]$.

After a finite succession of elementary transformations of type I and II, we can assume $\sigma = 0$ or $\sigma = 1$ or $\sigma \geq 2\nu_1$ and moreover if $B = 0$, then we assume that $\sigma \geq 2\nu_1$ and $\sigma \leq e$.

3.2 III(p, ν_1)

In the case when $B = 1$, we get $\Delta_\infty^2 = -1$; hence $\Delta_\infty$is also an exceptional curve. Take a point p from $S - \Delta_\infty$ with multiplicity ν_1 and blow up at p. Then we obtain a non-singular surface U and a proper birational morphism $\mu : U \to \Sigma_1$. The inverse image of p is an exceptional curve E, that satisfies $\Delta_\infty \cap E = \emptyset$. Letting C' denote the proper inverse image of C, we get

$$C' \sim \sigma\Delta_\infty + e(F' + E) - \nu_1 E.$$

Contracting Δ_∞ into a non-singular point q, we get a non-singular surface W and a proper birational morphism $\lambda : U \to W$. W is isomorphic to Σ_1, which has a $\mathbf{P}^1-$ fibering. The image of E is a section of the fibering, which we denote by Δ. Then $\Delta^2 = -1$. The image C_0 of C' by λ is written as follows for some σ' and e' in the space of linear equivalence classes:

$$C_0 \sim \sigma'\Delta + e'F_v.$$

Here F_v denotes a general fiber of the $\mathbf{P}^1-$ bundle of W. By the same argument as before, we get

$$\sigma' = e - \nu_1, \quad e' = e, \quad \nu_1' = e - \sigma,$$

where ν_1' indicates the multiplicity of C_0 at q. The birational map φ : $W \to \Sigma_1$ that is a composition of μ and λ^{-1} is said to be an *elementary transformation* $\mathrm{III}(p, \nu_1)$. Then

$$C^2 - {\nu_1}^2 = \sigma(2e - \sigma) - {\nu_1}^2$$

and

$${C_0}^2 - {\nu_1'}^2 = (e - \nu_1)(2e - e + \nu_1) - (e - \sigma)^2 = \sigma(2e - \sigma) - {\nu_1}^2 = C^2 - {\nu_1}^2.$$

Letting (S, D) be a pair obtained by minimal resolution of singularities from the pair (Σ_B, C) and (S_0, D_0) a pair obtained by minimal resolution of singularities from the pair by the elementary transformation $\mathrm{III}(p, \nu_1)$.

If $\nu_1' \neq 1$, then

$${D_0}^2 - D^2 = 0$$

and moreover, if $\nu_1' = 1$, then

$${D_0}^2 - D^2 = 1.$$

Also in these cases, we write ${D_0}^2 - D^2 = \varepsilon(\mathrm{III}(p, \nu_1))$.

Now we take a point p_1 where $\nu_1 = \mathrm{mult}_{\,p_1}(C) = \nu_1(C)$. If $e - \sigma < \nu_1$, then Δ_∞ does not pass through p_1, since $e - \sigma = \Delta_\infty \cdot C < \mathrm{mult}_{p_1}(C) = \nu_1$. Thus we can apply an elementary transformation of type III with center p_1 and then the transformed curve C_0 has the type $[\sigma' * e', 1; \nu_1', \nu_2, \cdots, \nu_r]$, where $\nu_1' = e - \sigma < \nu_1$ and $\sigma' = e - \nu_1 < \sigma$. Note that ν_1' may be smaller than ν_2.

3.3 $\#-$ minimal model

Finally we consider the case when C is itself non-singular. If $B = 1$ and $e - \sigma = \nu_1 = 1$, then Δ_∞ is an exceptional curve with $\Delta_\infty \cdot C = 1$. This implies that (Σ_1, C) is not relatively minimal. Contracting Δ_∞ into a non-singular point of $\mathbf{P}^2$, we get a non-singular curve C_1 on $\mathbf{P}^2$. The contraction gives rise to a birational morphism $\lambda : \Sigma_1 \to \mathbf{P}^2$ which is the inverse of the blowing up. The morphism λ is said to be *a transformation* $\mathrm{O}_-(\Delta_\infty)$.

Definition 2 *Assume that $\sigma \geq 2\nu_1$ and $e \geq \sigma + B\nu_1$. Moreover, if $B = 1$ then assume $e - \sigma > 1$. When the above conditions are satisfied, the pair (Σ_B, C) is said to be $\#-$ minimal. Occasionally, the $\#-$ minimal pair (Σ_B, C) is said to be a $\#-$ minimal model of a pair (S, D), if it is birationally equivalent to (S, D).*

For simplicity, the curve C is said to be $\#-$ minimal, whenever the pair (Σ_B, C) is $\#-$ minimal.

4 logarithmic plurigenera

Let (S, D) be a non-singular minimal pair. Then either $S = \mathbf{P}^2$ or (S, D) is derived from a $\#-$ minimal pair (Σ_B, C) of type $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$ by a finite succession of blowing ups at singular points of C.

The next relations among linear equivalence classes hold:

$$D \sim C - \sum_{j=1}^{r} \nu_j E_j, \quad K_S \sim K_0 + \sum_{j=1}^{r} E_j,$$

$$D + \nu_1 K_S \sim C + \nu_1 K_0 + \sum_{j=1}^{r} (\nu_1 - \nu_j) E_j.$$

Then $|C + \nu_1 K_0| \neq \emptyset$ and so $|D + \nu_1 K_S| \neq \emptyset$.

4.1 nef divisors

We recall some basic results on non-singular minimal pairs (S, D) under the assumption $g = g(D) > 0$ ([6, Iitaka]). For simplicity, $g - 1$ is denoted by $\overline{g}$.

Whenever $g > 0$, $Z = K_S + D$ is a nef divisor and $Z \cdot D = 2\overline{g}, Z^2 \geq 0$. Moreover, $Z^2 = 0$ if and only if $\kappa[D] = 0, 1$ ([6, Proposition 3,p299]).

We shall prove the following three lemmas.

Lemma 1 *1.* $(D + \nu_1 K_S) \cdot D \geq 0$ *,*

2. $(\nu_1 - 1)D^2 \leq 2\nu_1 \overline{g}$.
In particular, if $\nu_1 \geq 2$ *then* $D^2 \leq 4\overline{g}$. *Moreover, if* $\nu_1 \geq 3$ *then* $D^2 \leq 3\overline{g}$.

Proof: Suppose that $(D + \nu_1 K_S) \cdot D < 0$. Then since $|D + \nu_1 K_S| \neq \emptyset$, it follows that $|D+\nu_1 K_S - D| \neq \emptyset$. This implies that $|\nu_1 K_S| \neq \emptyset$; hence, $\kappa(S) \geq 0$. This contradicts that S is a rational surface. Hence, $(D + \nu_1 K_S) \cdot D \geq 0$. On the other hand, $(\nu_1 Z - (\nu_1 - 1)D) \cdot D = 2\nu_1 \overline{g} - (\nu_1 - 1)D^2$, which induces the result. $\square$

Lemma 2 *Suppose that* $\nu_1 \geq \nu > 0$. *If* Y *is a nef divisor on* S*, then* $(D + \nu K_S) \cdot Y \geq 0$.

Proof: Since $|D + \nu_1 K_S| \neq \emptyset$, taking F from $|D + \nu_1 K_S|$ we obtain

$$\nu_1(D + \nu K_S) \sim \nu_1 D + \nu(F - D) = (\nu_1 - \nu)D + \nu F.$$

Hence, $\nu_1(D + \nu K_S) \cdot Y = ((\nu_1 - \nu)D) \cdot Y + \nu F \cdot Y \geq 0$; thus we obtain the result. $\square$

Lemma 3 (Theorem of adjoint of special index 2) *Under the hypothesis that* $\kappa[D] = 2$ *and* $\sigma \geq 4$, $2Z - D = D + 2K_S$ *is a nef divisor. Moreover,* $(2Z - D)^2 \geq 0$. *If* $(2Z - D)^2 = 0$ *then* $2Z - D \sim 0$ *and* $\sigma = 4$.

When the type is $[d; 1]$, $2Z - D$ *is a nef divisor if and only if* $d \geq 6$. *Moreover, if* $(2Z - D)^2 = 0$ *then the type is* $[6; 1]$.

Proof: Since $\sigma \geq 4$, it follows that $2Z_0 - C \sim (\sigma - 4)\Delta_0 + (f + 2B - 4)F_c$, which is nef. Moreover, $(2Z_0 - C)^2 = (\sigma - 4)(B\sigma + 2f - 8) \geq 0$. Thus $(2Z_0 - C)^2 = 0$ if and only if $\sigma = 4$. Therefore, if $\nu_1 = 1$, namely if C is nonsingular, the result follows.

Suppose that $\nu_1 \geq 2$. Even in the case where $g = 0$, $|D + 2K_S| \neq \emptyset$.

Assume that there were an irreducible curve A such that $(2Z - D) \cdot A < 0$. Then $D \neq A$. Indeed, by Lemma 1, $D^2 \leq \dfrac{2\nu_1}{\nu_1 - 1}\overline{g} \leq 4\overline{g}$. In particular, $(D + 2K_S) \cdot D = (2Z - D) \cdot D = 4\overline{g} - D^2 \geq 0$.

Taking F from $|D + \nu_1 K_S|$, we get

$$\nu_1(2Z - D) \sim \nu_1 D + 2\nu_1 K_S \sim 2F + (\nu_1 - 2)D.$$

Then $A\cdot(2F+(\nu_1-2)D)<0$ and so $2A\cdot F<-A(\nu_1-2)\cdot D\leq 0$, which would imply that $A\cdot F<0$ and so $A^2<0$. Moreover,

$$0>(2Z-D)\cdot A=D\cdot A+2K_S\cdot A\geq 2K_S\cdot A.$$

Therefore, $A^2=A\cdot K_S=-1$ and so $0>D\cdot A-2$; thus $D\cdot A<2$. But since (S,D) is minimal, it follows that $A\cdot D\geq 2$; a contradiction.

Recalling that $\nu_1\geq 2$ and that $2Z-D$ is nef, by Lemma 2 we get $(2Z-D)^2\geq 0$.

Assume that $(2Z-D)^2=0$. We shall examine the equality in the following cases, separately.

case (1) $\nu_1\geq 3$.

Since $2Z-D$ is nef and $\nu_1\geq 3$, it follows that $(3Z-2D)\cdot(2Z-D)\geq 0$. But

$$\begin{aligned}0\leq&(3Z-2D)\cdot(2Z-D)\\=&2(2Z-D)\cdot(2Z-D)+-Z\cdot(2Z-D)\\=&-Z\cdot(2Z-D)\leq 0.\end{aligned}$$

Hence, $(3Z-2D)\cdot(2Z-D)=Z\cdot(2Z-D)=0$. Therefore, $D\cdot(2Z-D)=0$; hence it follows that $D^2=4\overline{g}$ and $Z^2=\overline{g}$.

Assume that $g\geq 1$. Then Z is nef and big. Hence, $D+2K_S=2Z-D\sim 0$ by Hodge's index Theorem.

Moreover,

$$\nu_1Z-(\nu_1-1)D\sim\nu_1Z-2(\nu_1-1)Z=(2-\nu_1)Z.$$

But $|\nu_1Z-(\nu_1-1)D|\neq\emptyset$ and $\kappa(S,Z)\geq 0$. Hence, $Z\sim 0$, which contradicts $\kappa(S,Z)=\kappa[D]=2$.

Assume that $g=0$. Then $D^2=4\overline{g}=-4$, which contradicts the fact $D^2\leq -5$.

case (2) $\nu_1\leq 2$.

Since $2Z-D\sim D+2K_S\sim C+2K_0$, it follows that

$$(2Z-D)^2=(C+2K_0)^2=(\sigma-4)(\sigma B+2f-8)\geq 0.$$

But $\sigma\geq 4$ and $Q=0$. Hence, we obtain $\sigma=4$. $\square$

In what follows, Q stands for $(2Z-D)^2$.

4.2 formula for plurigenera

Proposition 2 *Suppose that* (S, D) *is minimal with* $g = g(D) \geq 2$ *and* $\kappa[D] = 2$. *Then letting* $Z = K_S + D$, *for any* $m > 0$, *if* $g > 1$ *then*

$$P_m[D] = \frac{m(m-1)}{2} Z^2 + m\overline{g} + 1,$$
$$P_2[D] = Z^2 + 2g - 1 = Z^2 + 2\overline{g} + 1.$$

Moreover, if $g = 1$ *then*

$$P_m[D] = \frac{m(m-1)}{2} Z^2 + 2,$$
$$P_2[D] = Z^2 + 2.$$

Proof: Since Z is nef and big, by a vanishing theorem due to Kawamata, $H^1(S, \mathcal{O}_S(K_S + mZ)) = 0$ for any $m > 0$. Hence, by Riemann-Roch,

$$\dim H^0(S, \mathcal{O}_S(K_S + mZ)) = \frac{mZ \cdot (K_S + mZ)}{2} + 1 = \frac{m(m+1)}{2} Z^2 - \overline{g}m + 1.$$

From the exact sequence of sheaves

$$0 \to \mathcal{O}_S(K_S + mZ) \to \mathcal{O}_S((m+1)Z) \to \mathcal{O}_D((m+1)K_D) \to 0$$

we obtain

$$P_{m+1}[D] = \dim H^0(S, \mathcal{O}_S(K_S + mZ)) + \dim H^0(D, \mathcal{O}_D((m+1)K_D))$$

If $m \geq 2$ and $g > 1$ then $H^0(D, \mathcal{O}_D((m+1)K_D) = 0$;hence,

$$P_{m+1}[D] = \frac{m(m+1)}{2} Z^2 + \overline{g}(m+1) + 1.$$

If $m \geq 2$ and $g = 1$ then $H^0(D, \mathcal{O}_D((m+1)K_D) = \mathbf{C}$;hence,

$$P_{m+1}[D] = \frac{m(m+1)}{2} Z^2 + 2,$$
$$P_2[D] = Z^2 + 2.$$

Here,

$$Z^2 = (K_S + D)^2 = 4\overline{g} + K_S^2 - D^2.$$

□

Later, it will be shown that if $g = 0$ then $P_2[D] = Z^2 + 2$.
Note that $P_2[D]$ may be called *bigenus.*

We shall show some relations among $Z^2, Z \cdot D, D^2$ involving the multiplicities of singularities.

4.3 Formula I

Let (S, D) be a non-singular minimal pair which is birationally equivalent to a # minimal pair (Σ_B, C) of type $[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$. Let Z_0 denote $C + K_0$ and let t_j denote the number of $j-$ ple singular points of the curve C. For simplicity, by $\tilde{B}$ we denote $B\sigma + 2f$.

Definition 3 *Define τ_m to be $(\sigma - m)(\tilde{B} - 2m)$.*

For example, $\tau_1 = (\sigma - 1)(\tilde{B} - 2) = 2g_0$.

Lemma 4 *For any integers ν, μ,*

$$(\nu Z_0 - (\nu - 1)C) \cdot (\mu Z_0 - (\mu - 1)C) = \tau_{\nu+\mu} - 2(\nu - \mu)^2.$$

In particular, $(\nu Z_0 - (\nu - 1)C) \cdot Z_0 = \tau_{\nu+1} - 2(\nu - 1)^2$ and $(\nu Z_0 - (\nu - 1)C) \cdot (2Z_0 - c) = \tau_{\nu+2} - 2(\nu - 2)^2$.

Proof: By definition,

$$\begin{aligned}
&(\nu Z_0 - (\nu - 1)C) \cdot (\mu Z_0 - (\mu - 1)C) \\
=&((\sigma - 2\nu)\Delta_0 + (f + \nu B - 2\nu)F_c)((\sigma - 2\mu)\Delta_0 + (f + \mu B - 2\mu)F_c) \\
=&(\sigma - 2\nu)(\sigma - 2\mu)B + (\sigma - 2\nu)(f + \mu B - 2\mu) + (\sigma - 2\mu)(f + \nu B - 2\nu) \\
=&(\sigma - \nu - \mu)B\sigma + (\sigma - 2\nu - 2\mu)(f - 2\mu) + (\sigma - 2\nu - 2\mu)(f - 2\nu) \\
&+ (\mu - \nu)(f - 2\nu) - (\mu - \nu)(f - 2\mu) - 2(\mu - \nu)^2 \\
=&(\sigma - \nu - \mu)(B\sigma + 2f - 2\nu - 2\mu) - 2(\mu - \nu)^2 \\
=&\tau_{\nu+\mu} - 2(\nu - \mu)^2.
\end{aligned}$$

□

From $(\nu Z_0 - (\nu - 1)C) \cdot (2Z_0 - C) - 2(\nu Z_0 - (\nu - 1)C) \cdot Z_0 = -(\nu Z_0 - (\nu - 1)C) \cdot C$ and Lemma 4, we get the next result, which would be very useful.

Lemma 5 (Formula I)

1. *Letting $\widetilde{\delta}(\nu)$ be $\sum_{j=2}^{\nu_1} (j - 1)(\nu - j)t_j$, we obtain*

$$(\nu Z - (\nu - 1)D) \cdot Z = (\nu Z_0 - (\nu - 1)C) \cdot Z_0 + \widetilde{\delta}(\nu),$$

$$(\nu Z_0 - (\nu - 1)C) \cdot Z_0 = \tau_{\nu+1} - 2(\nu - 1)^2.$$

2. Letting $\widetilde{\delta}_0(\nu)$ *be* $\sum_{j=2}^{\nu_1} j(\nu-j)t_j$*, we obtain*

$$(\nu Z - (\nu-1)D)\cdot D = (\nu Z_0 - (\nu-1)C)\cdot C + \widetilde{\delta}_0(\nu),$$
$$(\nu Z_0 - (\nu-1)C)\cdot C = \tau_\nu - 2\nu^2.$$

3. Letting $\widetilde{\delta}_1(\nu)$ *be* $\sum_{j=2}^{\nu_1}(\nu-j)^2 t_j$*, we obtain*

$$(\nu Z - (\nu-1)D)^2 = (\nu Z_0 - (\nu-1)C)^2 - \widetilde{\delta}_1(\nu),$$
$$(\nu Z_0 - (\nu-1)C)^2 = \tau_{2\nu}.$$

By Lemma 4, the next result is obtained.

Corollary 1

$$\nu(\tau_{\mu+1} - 2(\mu-1)^2) - (\nu-1)(\tau_\mu - 2\mu^2) = \tau_{\nu+\mu} - 2(\nu-\mu)^2,$$

and

$$\nu\tau_{\mu+1} - (\nu-1)\tau_\mu = \tau_{\nu+\mu} - 2\nu^2 + 2\nu.$$

Remark 2 *When* $\nu_1 \leq 2$*,*

$$\widetilde{\delta}(2) = \widetilde{\delta}_0(2) = \widetilde{\delta}_1(2) = 0.$$

When $\nu_1 \leq 3$*,*

$$\widetilde{\delta}(3) = t_2, \quad \widetilde{\delta}_0(3) = 2t_2, \quad \widetilde{\delta}_1(3) = t_2.$$

Corollary 2 *When* $\nu_1 \leq 2$*,*

$$(\sigma-3)(B\sigma+2f-6) = 4 - 2g + 2Z^2.$$

Proof: Applying Remark in the case when $\nu_1 \leq 2$ and $\nu = 2$, we obtain

$$2Z^2 - 2g + 2 = (2Z - D)\cdot Z = \tau_3 - 2,$$

where $\tau_3 = (\sigma-3)(B\sigma+2f-6)$. $\square$

Claim 1 *Let* $X = \sigma - m$ *and* $Y = \tilde{B} - 2m$*. If* $\sigma \geq m$*, then* $X \leq Y$ *except for* $B = 1$ *and* $m > 2f$*.*

In the exceptional case, $B = 1$ and $m > 2f \geq 4$. Hence, $m \geq 5$.

4.4 mixed plurigenera

If $m \geq a$ then every $\dim|mK_S + aD| + 1$ is also a birational invariant as pairs, which is denoted by $P_{m,a}[D]$. They are called **mixed plurigenera.**

If $g > 0$ then $Z = K_S + D$ is nef and big. Hence, $H^1(S, \mathcal{O}_S(K_S + Z)) = 0$ by a vanishing theorem ;thus

$$P_{2,1}[D] = \dim H^0(S, \mathcal{O}_S(K_S + Z)) = \frac{(K_S + Z) \cdot Z}{2} + 1 = Z^2 - g + 2.$$

By Lemma 3, if $g > 0, \kappa[D] = 2$ and $\sigma \geq 5$ or $d \geq 7$, $2Z - D = D + 2K_S$ is a nef and big divisor. Hence, $H^1(S, \mathcal{O}_S(D + 3K_S)) = H^1(S, \mathcal{O}_S(K_S + 2Z - D)) = 0$;thus

$$\begin{aligned} P_{3,1}[D] = \dim H^0(S, \mathcal{O}_S(3K_S + D)) =& \frac{(3Z - 2D) \cdot (2Z - D)}{2} + 1 \\ =& 3A - \alpha + 1 \\ =& \frac{Q + 2A + D^2 - 4\overline{g}}{2} + 1 \\ =& 3Z^2 + 8 - 7g + D^2. \end{aligned}$$

Here $A = \frac{Z \cdot (2Z - D)}{2}$ and $\alpha = D \cdot (2Z - D) = 4\overline{g} - D^2$.

Therefore we obtain the next proposition:

Proposition 3 *Suppose that* $\kappa[D] = 2$ *and* $\sigma \geq 5$ *or* $d \geq 7$. *Then*

$$P_{3,1}[D] = 3Z^2 + 1 - 7\overline{g} + D^2 \geq 0.$$

Theorem 1 (Existence of adjoint of special index 3) *Assume that* $\sigma \geq 6$. *Then either* $|D + 3K_S| \neq \emptyset$ *or the type is* $[6 * 8, 1; 2^r]$.

Proof: Suppose that $P_{3,1}[D] = 0$. Then $3Z^2 + 1 - 7\overline{g} + D^2 = 0$, i.e., $(3Z - 2D) \cdot (2Z - D) + 2 = 0$. Then $\nu_1 \neq 3$ and we shall show that $\nu_1 \leq 2$. Actually, otherwise $\nu_1 \geq 4$ and so $|D + \nu_1 K_S| \neq \emptyset$. Letting Y be $D + 2K_S$, which is nef and big for $\sigma > 5$.

Taking F from $|D + \nu_1 K_S|$ we have

$$\nu_1(3Z - 2D) \sim 3(F - D) + \nu_1 D = (\nu_1 - 3)D + 3F.$$

By computing intersection numbers with Y we obtain

$$\nu_1(3Z - 2D) \cdot Y = (\nu_1 - 3)D \cdot Y + 3F \cdot Y \geq 0.$$

But $(3Z - 2D) \cdot Y = -2$. This is a contradiction. Therefore, $\nu_1 \leq 2$ has been established and so $\widetilde{\delta}(2) = \widetilde{\delta_0}(2) = 0$.

Letting $\tilde{B}$ be $B\sigma + 2f$, we obtain by a corollary to Lemma 4

$$\begin{aligned}(3Z - 2D) \cdot (2Z - D) =& 3Z \cdot (2Z - D) - 2D \cdot (2Z - D) \\ =& 3(\tau_3 - 2) - 2(\tau_2 - 8) \\ =& \tau_5 - 2 \\ =& (\sigma - 5) \cdot (\tilde{B} - 10) - 2.\end{aligned}$$

Hence, $(\sigma - 5) \cdot (\tilde{B} - 10) = 0$, which implies $\tilde{B} - 10 = 0$, i.e., $B\sigma + 2f = 10$. Therefore, $\sigma = 6, B = 1, f = 2$. $\Box$

In particular , $(D+3K_S) \cdot D \geq 0$ if $\sigma \geq 6$ except for the case of $[6*8, 1; 2^r]$ where $r = 0, 1, 2$. Indeed, in the case of $[6 * 8, 1; 2^r]$, one has $2\omega = (D + 3K_S) \cdot D = 2(r - 3)$.

Theorem 2 *Suppose that* $\sigma \geq 6$ *and* $g > 0, \kappa[D] = 2$ *where the type is not* $[6 * 8, 1; 2^r]$, $r = 0, 1, 2$. *If* $(D + 3K_S) \cdot D = 0$ *then either* $D + 3K_S \sim 0$ *or the type is* $[6 * 8, 1; 2^3]$.

Proof: First, under the assumption that the type is not $[6 * 8, 1; 2^r]$, we shall show that $3Z - 2D = D + 3K_S$ is nef. Actually, otherwise there exists an irreducible curve A such that $(D + 3K_S) \cdot A < 0$. From hypothesis $(D + 3K_S) \cdot D = 0$, we derive $A \neq D$; hence, $D \cdot A \geq 0$.

Since $|D + 3K_S| \neq \emptyset$, it follows that $A^2 < 0$ and so $(D + 3K_S) \cdot A = D \cdot A + 3K_S \cdot A < 0$. Therefore, A turns out to be an exceptional curve. But since (S, D) is minimal, we obtain $D \cdot A = 2$. Therefore, contracting A into a non-singular point p_0 on a nonsingular surface W, we obtain a proper birational morphism $\mu : S \to W$. Let D_0 be $\mu(D)$, which has a double point at p_0. Then $D \sim D_0 - 2A$ and $K_S \sim K_W + A$. Putting $Y = D + 3K_S, Y_0 = D_0 + 3K_W$, we obtain $Y \cdot D = 0, Y \sim Y_0 + A$; hence

$$Y \cdot D = (Y_0 + A) \cdot (D_0 - 2A) = Y_0 \cdot D_0 + 2.$$

Since $|Y| = |Y_0| + A$, it follows that $|Y_0| \neq \emptyset$. Hence, $Y_0 \cdot D_0 \geq 0$. Actually, otherwise, $Y_0 \cdot D_0 < 0$ implies that D_0 is a fixed component of $|Y_0|$ and thus $\emptyset \neq |Y_0| - D_0 = |3K_W|$, a contradiction. Therefore, $Y_0 \cdot D_0 \geq 0$. However, by hypothesis, $Y \cdot D = 0$ and by definition $-2 = Y \cdot D - 2 = Y_0 \cdot D_0 \geq 0$; a contradiction.

Therefore, $3Z - 2D$ is nef and so $(3Z - 2D)^2 \geq 0$.

If $(3Z-2D)^2>0$ then $(3Z-2D)\cdot D=0$ implies that $D\sim 0$ or $D^2<0$ by Hodge's index theorem . But $0\leq 6\overline{g}=3Z\cdot D=2D^2$; a contradiction. Hence, $(3Z-2D)^2=0$ has been established.

But $0=(3Z-2D)^2=3(3Z-2D)\cdot Z-2(3Z-2D)\cdot D=3(3Z-2D)\cdot Z$. Hence, $(3Z-2D)\cdot Z=0$. Recalling that Z is nef and big, we conclude that $3Z-2D=D+3K_S\sim 0$. $\square$

From the proof of Proposition 2, we derive the following formula:

Proposition 4 *Suppose that (S,D) is minimal with $g>0$ and $\kappa[D]=2$. Then*

$$P_{m,m-1}[D]=\frac{m(m-1)}{2}Z^2-\overline{g}(m-1)+1,$$
$$P_{2,1}[D]=Z^2-\overline{g}+1=Z^2-g+2.$$

Since $P_{2,1}[D]\geq 0$, it follows that $Z^2\geq\overline{g}-1$ and hence, $P_2[D]=Z^2+2\overline{g}+1=Z^2+2g-1\geq 3\overline{g}$.

Moreover, if $g>0, \kappa[D]=2$ and $\sigma>4$ then $W=\frac{3}{2}\times(2Z-D)$ is a nef and big divisor with fractional part. Since $\lceil W\rceil=3Z-D=3K_S+2D$, we derive $H^1(S,\mathcal{O}_S(K_S+3K_S+2D))=0$;thus

$$\begin{aligned}P_{4,2}[D]=&\dim H^0(S,\mathcal{O}_S(4K_S+2D))\\=&\frac{(4K_S+2D)\cdot(3K_S+2D)}{2}+1\\=&(2Z-D)\cdot(3Z-D)+1\\=&6Z^2-10\overline{g}+1+D^2.\end{aligned}$$

4.5 estimates for bigenera

Suppose that $\sigma\geq 4$. By Lemma 3, we get

$$0\leq(D+2K_S)\cdot Z=(2Z-D)\cdot Z=2Z^2-D\cdot Z=2(Z^2-g+1),$$
$$P_{2,1}[D]=Z^2+2-g.$$

Thus, if $g>1$,

$$Z^2\geq\overline{g}\ \text{ and }\ P_2[D]=Z^2+2g-1=\geq 3g-2.$$

Further, if $g > 1$,then

$$P_2[D] = Z^2 + 2g - 1 = P_{2,1}[D] + 3g - 3.$$

If $g = 1$ then

$$P_2[D] = Z^2 + 2 = P_{2,1}[D] + 1.$$

- Suppose that $S = \Sigma_B$ and $\sigma = 3$.
  Then $g \geq 4, D^2 = 3g + 6$ and $Z^2 = 8 - D^2 + 4g - 4 = g - 2$.

- Suppose that $\nu_1 = 1$ and $S = \mathbf{P}^2$. If the type is $[d; 1]$ where $d \geq 4$, then $Z = D + K_S \sim (d-3)H$, H being a line.
  Since $Z^2 = (d-3)^2$ and $g_0 = \dfrac{(d-1)(d-2)}{2}$, it follows that

$$Z^2 - (g_0 - 2) = \frac{(d-4)(d-5)}{2} \geq 0.$$

Consequently, we obtain the following result.

Theorem 3 *Suppose that (S, D) is a relatively minimal pair with $g = g(D) \geq 1$. Letting Z be $K_S + D$, we obtain*

1. *If $g > 1$ then $P_2[D] = Z^2 + 2g - 1 \geq 2g - 1$.*
2. *If $g > 1$ and $P_2[D] = 2g - 1$ or $g = 1$ and $P_2[D] = 2$, then $Z^2 = 0$ and $\kappa[D] = 0$ or 1.*
3. *If $\kappa[D] = 2, g > 1$, then $Z^2 \geq g - 2$ and $P_2[D] \geq 3g - 3$.*
4. *If $\kappa[D] = 2, g = 1$, then $Z^2 \geq 1$ and $P_2[D] = Z^2 + 2 \geq 3$.*
5. *If $P_2[D] = 3g - 3$ and $g > 2$, then $Z^2 = g - 2$ and one of the following cases occurs.*
   - *(a) $S = \Sigma_B$ and $\sigma = 3$ or*
   - *(b) $S = \mathbf{P}^2$ and $d = 4$ or 5.*
     In both cases, $P_{2,1}[D] = 0$.
6. *$P_{21}[D] = A + 1$, where $A = Z^2 - \overline{g}$.*
7. *If $g > 1$ then $P_2[D] = P_{21}[D] + 3\overline{g}$*

In that follows, we shall determine types of #minimal pairs of minimal pairs (S, D) with $P_2[D] = 2g, 2g+1; 3g-2, 3g-1, 3g$, in other words, (S, D) with $Z^2 = 1, 2; g-1, g, g+1$.

First, we consider the case in which the type is $[d; 1]$ where $P_2[D]$ is small.

Proposition 5 *Assume that the type is* $[d; 1]$.

1. *If* $Z^2 = 1$ *then* $d = 4$ *and* $P_2[D] = 6$.
2. *Assume that* $Z^2 = g - 2 + j$ *where* $j = 0, 1, 2, 3$.
   If $j = 0$ *then* $d = 4, 5$. *If* $j = 2$ *then* $d = 3, 6$. *If* $j = 3$ *then* $d = 4, 7$.

Proof: If the type is $[d; 1]$, then $Z^2 = (d-3)^2$. Assume that $Z^2 = 1$ or 2. Then $d = 4$ and $Z^2 = 1$.

Assume that $Z^2 = g - 2 + j, j \geq 0$. Then since $2g - 2 = (d-1)(d-2)$, it follows that $(d-3)(d-6) = j - 2$. Hence, the result follows immediately.

$\square$

5 relations between A and α

Two more invariants A, α are introduced:

$$A = (2Z - D) \cdot Z/2 = Z^2 - \overline{g}, \alpha = (2Z - D) \cdot D = 4\overline{g} - D^2.$$

Since $2Z - D$ is nef for $\sigma \geq 4$ and $\kappa[D] = 2$, both A and α are non-negative.

Proposition 6 *Suppose that a minimal pair* (S, D) *with* $\kappa[D] = 2$ *is derived from a # minimal pair* (Σ_B, C) *of type* $[\sigma * e, B; \nu_1, \cdots, \nu_r]$ *or* (S, D) *is just* $(\mathbf{P}^2, D)$ *of type* $[d; 1]$ *where* $d \geq 4$. *we shall show that the next relations between* A *and* α *hold.*

1. *When* $\sigma = 3$ *or* $d = 4, 5$*,it follows that* $A = -1$ *and* $\alpha \geq -10$.
2. *When* $\sigma = 4$ *or* $d = 6$*,it follows that* $4A = \alpha$.
3. *When* $\sigma = 5$ *or* $d = 7, 8$ *or the type is* $[6 * 8, 1; 2^r]$*,it follows that* $3A = \alpha - 1$.
4. *When* $\sigma \geq 6$ *where the type is not* $[6 * 8, 1; 2^r]$*, or* $d \geq 9$*, it follows that* $3A \geq \alpha \geq A$.

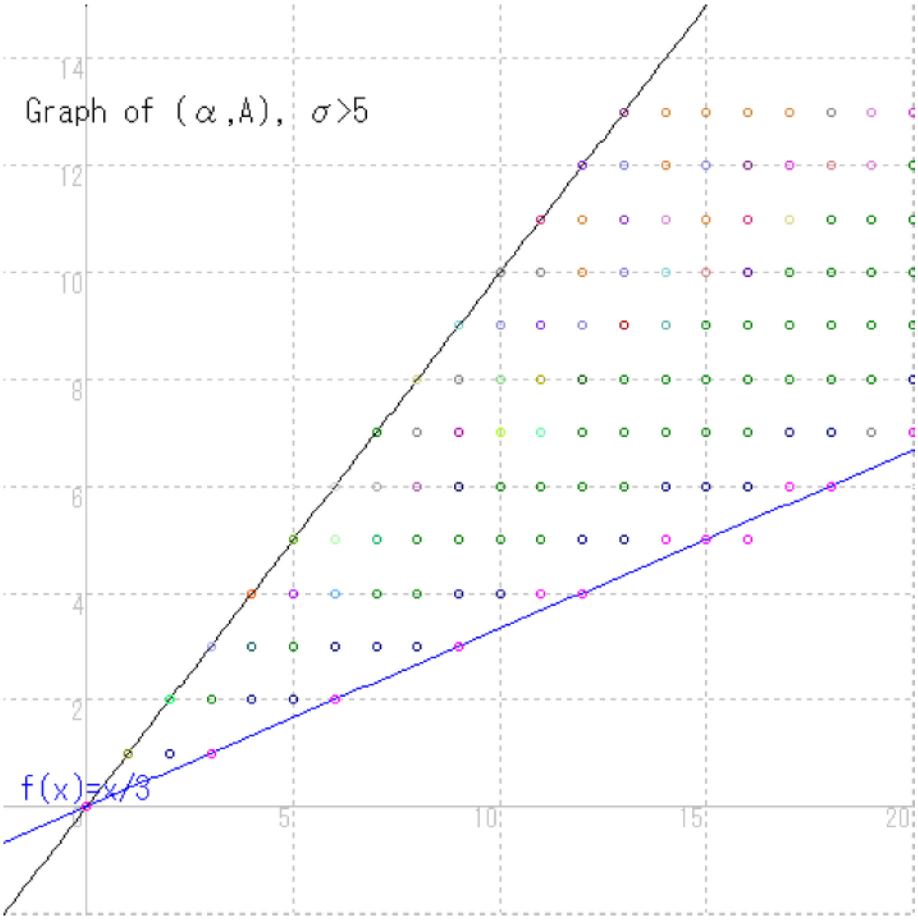

Figure 1: relations between α and A

Proof: First,we consider the case where (S, D) is a pair of the projective plane and a nonsingular curve D of which type is $[d; 1]$. Then $A = \frac{(d-3)(d-6)}{2}$ and $\alpha = d(d-6)$. Hence, $4A - \alpha = (d-6)^2, 3A - \alpha = \frac{(d-6)(d-9)}{2}$. From these, the assertion 1) follows.

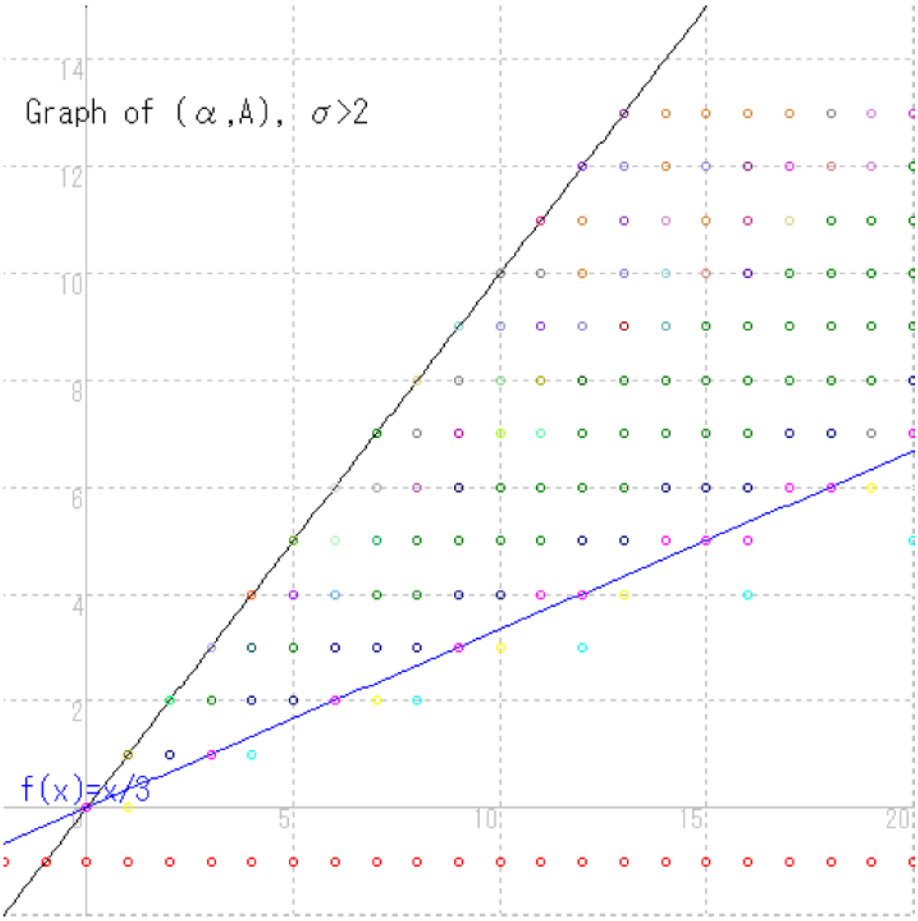

Figure 2: relations between α and A

Note that $4A - \alpha = Q = (2Z - D)^2$. Hence, by Lemma , $4A - \alpha = 0$ if and only if $\sigma = 4$.

If the type is $[6 * 8, 1; 2^r]$, then $\overline{g} = 19 - r, A = 5, \alpha = 16$.

Assume that $\sigma \geq 6$ and the type is not $[6 * 8, 1; 2^r]$. Then $3A - \alpha =$

$$\frac{(2Z-D)\cdot(3Z-2D)}{2} \geq 0.$$

We shall verify that $A \leq \alpha$ under the assumption $\sigma \geq 4$.

Since $K_S = Z - D$, it follows that $(Z-D)^2 = K_S^2$ and $Z^2 + D^2 - 4\overline{g} = K_S^2$. Moreover,

$$A - \alpha = Z^2 + D^2 - 5\overline{g} = K_S^2 - \overline{g}.$$

Case A): $K_S^2 \leq -1$.

Then

$$A - \alpha = K_S^2 - \overline{g} \leq -g \leq 0.$$

Case B): $K_S^2 \geq 0$.

By Riemann-Roch , $|-K_S| \neq \emptyset$. Hence, $(2Z-D)\cdot(D-Z) \geq 0$, which implies that $2Z^2 + D^2 - 6\overline{g} \leq 0$. Therefore,

$$A - \alpha = Z^2 + D^2 - 5\overline{g} \leq \overline{g} - Z^2 = -A \leq 0.$$

$\square$

Suppose that $\sigma \geq 4$ and $A - \alpha = 0$.

In case A): we get $g = 0, K_S^2 = -1$.There are many types in this case. But in case B), we get $g > 0, A = \alpha = 0$. Hence, the type is $[4*4; 2^r]*$ or $[6; 1]$.

6 relations between Ω and ω

Note that $\Omega \geq \omega$ when $\sigma \geq 6$ except for the type $[6*8, 1; 2^r]$. Indeed, except for the type $[6*8, 1; 2^r]$, since $|3Z - 2D| \neq \emptyset$ and $2Z - D$ is nef, we see that $(3Z-2D)\cdot(2Z-D) \geq 0$ and $(3Z-2D)\cdot(2Z-D) = 2(3Z-2D)\cdot Z - (3Z-2D)\cdot D = 2\Omega - 2\omega$.

6.1 Case $\nu_1 \leq 3$

Under the assumption that $\nu_1 \leq 3$ and $\sigma \geq 6$, we shall show that $\Omega \leq 3\omega$ provided that the type is not $[6*8, 1; 2^r]$.

By definition,

$$3\omega - \Omega = \frac{(\sigma-1)(\tilde{B}-2) - 50}{2} + 2t_2 = g_0 - 25 + 2t_2.$$

It is easy to check that $(\sigma-1)(\tilde{B}-2) \geq 50$, whenever the type is not $[6*8, 1; 2^r]$. Hence,

$$3\omega \geq \Omega.$$

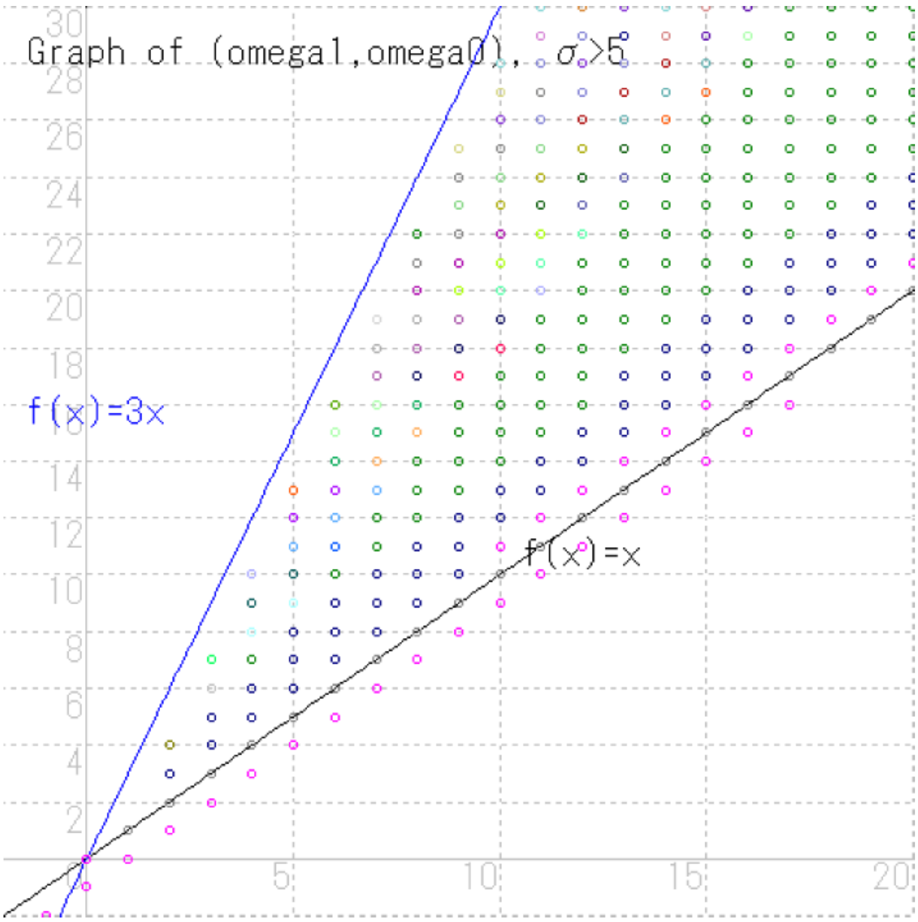

Figure 3: relations between ω and Ω

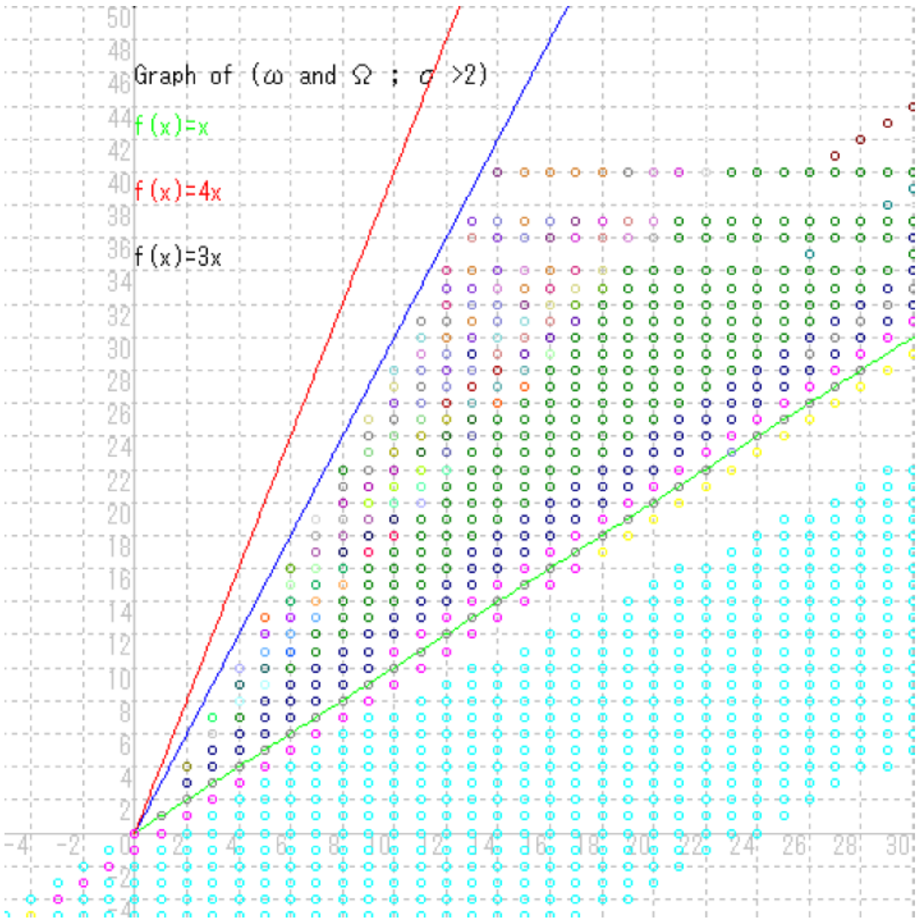

Figure 4: relations between ω and Ω, $\sigma > 2$

Note that if $3\omega = \Omega$ then the type is $[6 * 6; 3^{t_3}, 2^{t_2}]$. Hence, $\omega = \Omega = 0$. Except for these cases, $3\omega - \Omega \geq 2$.

Thus defining Υ to be $3\omega - \Omega$, we shall show that $\Upsilon \geq 2$.

6.2 Case $\nu_1 \geq 4$

From $K_S^2 = (Z - D)^2 = Z^2 - 4\overline{g} + D^2 = K_S^2 \leq -1$, it follows that

$$\Omega + \overline{g} - 3\omega = 3K_S^2 \leq -3.$$

Hence, $\Upsilon = 3\omega - \Omega = \overline{g} - 3K_S^2$.

We distinguish the following two cases:

Case A):$K_S^2 \leq -1$.
$\Upsilon = \overline{g} - 3K_S^2 \geq 2$.

Case B):$K_S^2 \geq 0$.
Then since $K_S^2 = r - 8$, it follows that $r \leq 8, g > 0$ and $|-K_S| \neq \emptyset$.

From $g_0 = (\sigma - 1)(\tilde{B} - 2)/2 \geq (2\nu_1 - 1)^2$, and $-\nu_j(\nu_j - 1) \geq -\nu_1(\nu_1 - 1)$, we get

$$\begin{aligned}\Upsilon = \overline{g} - 3K_S^2 =& \overline{g_0} - \sum_{j=1}^{r} \nu_j(\nu_j - 1)/2 - 3(8 - r)\\ \geq& 4\nu_1(\nu_1 - 1) - r\nu_1(\nu_1 - 1)/2 - 3(8 - r)\\ \geq& (8 - r)\nu_1(\nu_1 - 1)/2\\ \geq& 3(8 - r)\end{aligned}$$

Thus if $r < 8$ then $\Upsilon \geq 3$.

Suppose that $r = 8$, namely $K_S^2 = 0$. Then we shall show $\Upsilon = \overline{g} \geq 2$.

(1) If $\Upsilon = \overline{g} = 0$ then $0 = K_S^2 = D^2 + Z^2$. By Riemann-Roch, $|-K_S| \neq \emptyset$. Hence, $|D - Z| \neq \emptyset$. Since $2Z - D$ is nef,

$$(D - Z) \cdot (2Z - D) = -2Z^2 - D^2 \geq 0.$$

By $D^2 + Z^2 = 0, weget - Z^2 \geq 0$, a contradiction.

(2) If $\overline{g} = 1$ then $0 = K_S^2 = D^2 + Z^2 - 4$. Since $2Z - D$ is nef,

$$(D - Z) \cdot (2Z - D) = 6 - 2Z^2 - D^2 \geq 0.$$

By $D^2 + Z^2 = 4$, we get $2 - Z^2 \geq 0$. If $2 = Z^2$ then $(2Z - D) \cdot K_S = 0$. By Hodge's index theorem, we obtain $K_S^2 < 0$, a contradiction.

Consequently, $Z^2 = 1$. By $|D + \nu_1 K_S| \neq \emptyset$, we get

$$(D + \nu_1 K_S) \cdot Z = (\nu_1 Z - (\nu_1 - 1)D) \cdot Z \geq 0.$$

Since

$$(\nu_1 Z - (\nu_1 - 1)D) \cdot Z = \nu_1 Z^2 - 2(\nu_1 - 1)\overline{g} = (2 - \nu_1) \geq 0$$

it follows that $\nu_1 = 2$ and so $2 = g = g_0 - r = g_0 - 8$. Hence , $g_0 = 10$. But

$$(\sigma - 3)(\overline{B} - 6) = 4 - 2g + 2Z^2 = 2.$$

Hence, $\sigma = 4, \overline{B} = 8$. Thus $g_0 = (\sigma - 1)(\overline{B} - 2)/2 = 9$.

This contradicts $g_0 = 10$.

Combining the above argument, we establish the following result.

Proposition 7 *For minimal pairs (S, D) with $\kappa[D] = 2$ which are derived from # minimal models of type $[\sigma * e, B; \nu_1, \cdots, \nu_r]$ or which is $(\mathbf{P}^2, D)$ of type $[d; 1]$, the next relations between Ω and ω hold.*

1. *When $\sigma = 3$,it follows that $\Omega = -g - 4$ and $\omega = -9$.*
2. *When the type is $[d; 1]$ or ,it follows that $\Omega = (d - 3)(d - 9)$ and $\omega = \dfrac{d(d-9)}{2}$; $3\omega - \Omega = \dfrac{(d+6)(d-9)}{2}$.*
3. *When the type is $[6*8, 1; 2^r]$, then $\Omega = r - 4, \omega = r - 3, 4\omega - \Omega = 3r - 8$.*
4. *When $\sigma \geq 6$ where the type is not $[6 * 8, 1; 2^r]$ or $d \geq 9$, it follows that $\Upsilon = 3\omega - \Omega \geq 0$. Furthermore, if $\Upsilon = 0$ then $D + 3K_S \sim 0$ and $\omega = \Omega = 0$.*
5. *Under the above condition, if $D + 3K_S \not\sim 0$, then $\Upsilon \geq 2$.*

7 curves with $Z^2 = 1$

Second, we shall study pairs (S, D) such that $Z^2 = 1$ where (S, D) is derived from a # minimal pair (Σ_B, C). Then $Z^2 = 1$. Since $P_{2,1} = Z^2 + 2 - g = 3 - g$, we see that $1 \leq g \leq 3$.

If $g > 1$ then $P_2 = Z^2 + 2g - 1 = 2g + 1$ and If $g = 1$ then $P_2 = Z^2 + 2 = 3$.

7.1 case $g = 2, 3$

• If $g = 3$, then $P_2[D] = 2g = 6 = 3g - 3$ and by Theorem 3, $\sigma = 3$ or $d = 4$. If $\sigma = g = 3$, then $f = B = 1$; thus $e = 4, e - \sigma = 1$. Applying the transformation $\mathrm{O}_-(\Delta_\infty)$, the type becomes $[4;1]$, too. Hence, we conclude that the type of the transformed curve is $[4;1]$.

• If $g = 2$, then $\nu_1 \geq 2$ and so

$$0 \leq (D + \nu_1 K_S) \cdot Z = \nu_1 Z^2 - 2(\nu_1 - 1)\overline{g} = 2 - \nu_1.$$

Hence, $\nu_1 = 2$; thus $\nu_1 = 2$. Therefore,

$$K_S{}^2 = 8 - r, \quad g_0 = g + r, \quad D^2 = C^2 - 4r.$$

To determine the type, we use the invariant τ_m introduced in the former section.

Applying Corollary 2 to the case in which $Z^2 = 1$, we obtain

$$(\sigma - 3)(\tilde{B} - 6) = 6 - 2g.$$

When $g = 2$, from $(\sigma - 3)(\tilde{B} - 6) = 2$ it follows that:

$$\sigma - 3 = 1, \quad \tilde{B} - 6 = 2.$$

Thus $\sigma = 4$ and then $g_0 = 9, r = 7, K_S{}^2 = 1, D^2 = 4$. According to the value of $B = 0, 1, 2$, f becomes 4, 2, 0, respectively. Then the type becomes $[4 * 4; 2^7]$ or its associates.

7.2 case $g = 1$

If $g = 1$, then $Z \cdot D = 0$. Since $Z^2 = 1$, we get $Z \cdot (Z - K_S) = 0$, and so $Z \cdot K_S = Z^2 = 1$. Further, $Z^2 = K_S{}^2 - D^2 = 1$ implies $K_S{}^2 = 1 + D^2$. In this case, $P_2[D] = 3$.

Claim 2

$$D^2 \leq -2.$$

Actually, suppose that $D^2 \geq -1$. Then $K_S{}^2 = 1 + D^2 \geq 0$. By Riemann-Roch, $\dim | - K_S| \geq K_S{}^2 \geq 0$. Hence, $Z \cdot -K_S \geq 0$;thus $Z \cdot K_S \leq 0$. But by hypothesis, $1 = Z \cdot K_S$; a contradiction.

7.3 Formula II

For a # minimal pair (Σ_B, C), letting t_j denote the number of $j-$ ple singular points of the curve C, define

ρ_{ν_1} to be $(D+2K_S)\cdot(D+\nu_1 K_S)$ and ζ_{ν_1} to be $\sum_{j=3}^{\nu_1-1}(\nu_1-j)(j-2)t_j$. Then

$$\rho_{\nu_1} = (C+2K_0)\cdot(C+\nu_1 K_0)+\zeta_{\nu_1}.$$

Since

$$\rho_{\nu_1} = (D+2K_S)\cdot D+\nu_1(D+2K_S)\cdot K_S$$

it follows that $(D+2K_S)\cdot D=(2Z-D)\cdot D=4\overline{g}-D^2$, which we denote by α and that $(D+2K_S)\cdot K_S/2 = D\cdot(Z-D)/2+K_S^2=\overline{g}-D^2/2+8-r$, which we denote by ξ_0 and hence, $\rho_{\nu_1}=2\nu_1\xi_0+\alpha$.

Replacing $B\sigma+2f$ by $\tilde{B}$ and $\sigma-2\nu_1$ by p where $p\geq 0$,respectively, we obtain

$$(C+2K_0)\cdot(C+\nu_1 K_0) = (C+2K_0)\cdot(C+\frac{\sigma-p}{2}K_0)$$

$$= (C+2K_0)\cdot(C+\frac{\sigma}{2}K_0)-\frac{p}{2}(C+2K_0)\cdot K_0).$$

Since

$$C+2K_0 \sim (\sigma-4)\Delta_0+(f+2B-4)F_c,$$

$$C+\frac{\sigma}{2}K_0 \sim (\frac{\tilde{B}}{2}-\sigma)F_c$$

it follows that

$$(C+2K_0)\cdot(C+\frac{\sigma}{2}K_0) = (\sigma-4)(\frac{\tilde{B}}{2}-\sigma),$$

which is denoted by $-\eta$, and that

$$(C+2K_0)\cdot K_0 = 16-2\sigma-\tilde{B}.$$

Thus letting $\tilde{\sigma}$ be $\sigma+\frac{\tilde{B}}{2}-8$, we obtain $(C+2K_0)\cdot(C+\nu_1 K_0)=-\eta+\tilde{\sigma}p$ and therefore,

$$\zeta_{\nu_1} = (D+2K_S)\cdot(D+\nu_1 K_S)-(C+2K_0)\cdot(C+\nu_1 K_0)$$

$$= \eta+2\nu_1\xi_0+\alpha-\tilde{\sigma}p = \eta+\sigma\xi_0+\alpha-(\xi_0+\tilde{\sigma})p.$$

Letting

$$\xi_2 = \xi_0+\tilde{\sigma} = \sigma+f-8+\frac{B\sigma}{2}+\xi_0,$$

we get

Proposition 8

$$\zeta_{\nu_1} = \eta + \sigma\xi_0 + \alpha - \xi_2 p.$$

Corollary 3 *Assume that* $\sigma \geq 4$.

1. *If* $B \neq 1$ *then* $\eta = (\sigma - 4)(\sigma - f - \frac{B\sigma}{2}) \leq 0$.
2. *If* $B = 1$ *then* $\eta \leq \frac{(\sigma-4)p}{2}$.

Moreover, $\eta - \xi_2 p \leq (\frac{D^2}{2} + r - g - 1 - f - \sigma)p$.

Proof of 2): Since $f \geq \nu_1 = \frac{\sigma - p}{2}$, it follows that $\sigma - f - \frac{B\sigma}{2} = \frac{\sigma}{2} - f \leq \frac{p}{2}$. Hence, $\eta \leq \frac{(\sigma-4)p}{2}$. □

7.4 sharper estimate

Letting $\tilde{\eta} = \eta - \tilde{\sigma}p$, we get

$$\begin{aligned}\eta &= (\sigma - 4)(\sigma - \frac{\tilde{B}}{2}) \\ &= 2(\nu_1 - 2)(\sigma - \frac{\tilde{B}}{2}) + p(\sigma - \frac{\tilde{B}}{2}) \\ &= 2(\nu_1 - 2)(2\nu_1 - \frac{\tilde{B}}{2}) + p(\sigma - \frac{\tilde{B}}{2}) + 2p(\nu_1 - 2)\end{aligned}$$

and

$$\begin{aligned}\tilde{\eta} = \eta - \tilde{\sigma}p &= 2(\nu_1 - 2)(2\nu_1 - \frac{\tilde{B}}{2}) + p(4 + 2\nu_1 - \tilde{B}) \\ &= -2(\nu_1 - 2)\gamma_1 + \tilde{A}p,\end{aligned}$$

where $\gamma_1 = (B-2)\nu_1 + f$ and $\tilde{A} = 2f - 4 - 2\nu_1 + B(\sigma + \nu_1 - 2)$. Then we obtain

$$\tilde{\eta} = -2(\nu_1 - 2)\gamma_1 - \tilde{A}p \leq -\tilde{A}p.$$

Now assume that $p \geq 1$ and $\nu_1 \geq 3$. Then since $\tilde{A} = B(\nu_1 + \sigma - 2) - 4 - 2\nu_1 + 2f$, it follows that

1. if $B = 0$ then $\tilde{A} = -4 - 2\nu_1 + 2f \geq -4 - 2\nu_1 + 2\sigma = -4 + 2\nu_1 + 2p \geq 2\nu_1 - 2$.

2. If $B = 1$ then $\tilde{A} \geq -4 - 2\nu_1 + 2f + 2 + 3\nu_1 + p \geq 3\nu_1 - 5 \geq 2\nu_1 - 3$.

3. If $B \geq 2$ then $\tilde{A} \geq 6 - 4\nu_1 = -4 + 2\nu_1 - 2 + 2\nu_1 \geq 4\nu_1 - 6 \geq 2\nu_1 - 2$.

Furthermore, since $p \geq 1$, it follows that when $B = 0$, $\gamma_1 = -2\nu_1 + f \geq -2\nu_1 + \sigma = p \geq 1$ and hence, $\tilde{\eta} \leq -2(\nu_1 - 2) - \tilde{A}p \leq 6 - 4\nu_1$.

Hence, we get the following estimate:

Lemma 6 *If $p \geq 1$ and $\nu_1 \geq 3$,then*

$$\zeta_{\nu_1} = \tilde{\eta} + 2\nu_1\xi_0 + \alpha$$

and

$$\begin{aligned}\tilde{\eta} &\leq (6 - 4\nu_1 + (\nu_1 - 1)\delta_{1,B})p, \\ &\leq (2 + \delta_{1,B} - 2\nu_1)p.\end{aligned}$$

7.5 case $D^2 = -2, -3, -4$

Using the formula above, we shall determine the type of pairs (S, D) in the case when $D^2 = -2, -3, -4$, examining the following cases, separately.

• case $\nu_1 \leq 2$. Since $(2Z - D) \cdot Z = 2Z^2 - D \cdot Z = 2Z^2$ and $(2Z - D) \cdot Z = \tau_3 - 2$, it follows that

$$\tau_3 = (\sigma - 3)(B\sigma + 2f - 6) = 2Z^2 + 2.$$

Since $Z^2 = 1$it follows that $2Z^2 + 2 = 4$ and $\sigma - 3 = 2$ or 1.

1. If $\sigma = 5$, then $2f + 5B - 6 = 2$, which is impossible.

2. If $\sigma = 4$, then $2f + 4B - 6 = 4$; hence, $\tilde{B} = 2f + 4B = 10$ and thus $g_0 = (\sigma - 1)(\tilde{B} - 2)/2 = 12$. This implies that $r = 11, D^2 = 2 \cdot 4 \cdot 5 - 4 \cdot 11 = -4$ and the type is $[4 * 5; 2^{11}]$ or its associates where $D^2 = 40 - 44 = -4$.

• case $\nu_1 = 3$. Then $|D + 3K_S| \neq \emptyset$ and so $(D + 3K_S) \cdot (D + 2K_S) \geq 0$. But

$$0 \leq (D + 3K_S) \cdot (D + 2K_S) = (3Z - 2D) \cdot (2Z - D) = 6 + 2D^2.$$

Hence, $D^2 \geq -3$.

When $D^2 = -3$, we get $(D + 3K_S) \cdot (D + 2K_S) = 0$.
Since $\nu_1 = 3$, it follows that

$$(D + 3K_S) \cdot (D + 2K_S) = (C + 3K_0) \cdot (C + 2K_0) = \tau_5 - 2.$$

From $(\sigma - 5)(\tilde{B} - 10) = \tau_5 = 2$, we obtain $\sigma - 5 = 1$ and $\tilde{B} - 10 = 2$. Hence, $\sigma = 6$ and $B\sigma + 2f = \tilde{B} = 12$. Therefore, the type is $[6 * 6; 3^{t_3}, 2^{t_2}]$ or their associates. Thus, the virtual genus $g_0 = 25$ and by genus formula

$$t_2 + t_3 = 10, \quad t_2 + 3t_3 = g_0 - g = 24.$$

Hence, $t_2 = 3, t_3 = 7$ and the the type is $[6 * 6; 3^7, 2^3]$ or its associates.
The case when $D^2 = -2$ will be treated in the next section.

• case $\nu_1 \geq 4$ or $D^2 = -2$.

Proposition 9 *If* $\nu_1 \geq 4, Z^2 = 1$ *and* $g = 1$*, then* $Z \cdot K_S = 1, D^2 = -2, r = 9$ *and* ${K_S}^2 = -1$.

Proof: By hypothesis, $Z \cdot K_S = Z^2 - Z \cdot D = 1 - 2\overline{g} = 1$. From

$$0 \leq (2Z - D) \cdot (\nu_1 Z - (\nu_1 - 1)D) = 2\nu_1 + (\nu_1 - 1)D^2,$$

it follows that $D^2 \geq -\dfrac{2\nu_1}{\nu_1 - 1} \geq -\dfrac{8}{3}$. Hence, $D^2 \geq -2$. By the Claim, $D^2 = -2$ is derived. Hence, from $D \cdot (D + K_S) = D^2 + D \cdot K_S = 0$, it follows that $D \cdot K_S = 2$. Moreover, $Z \cdot K_S = 1$ implies $1 = Z \cdot K_S = D \cdot K_S + K_S^2 = 2 + K_S^2$; hence, ${K_S}^2 = -1$ and $r = 9$. $\square$

In that follows , we assume $\nu_1 \geq 3$ and $D^2 = -2, r = 9$. Hence, $\xi_0 = 0, \alpha = 2$. Assume that $p \geq 1$. Then by a sharper estimate,

$$0 \leq \zeta_{\nu_1} = 2 + \tilde{\eta} \leq 2 + 5 - 3\nu_1 = 7 - 3\nu_1 \leq -2.$$

This is a contradiction. Therefore, $p = 0$ and $0 \geq \zeta_{\nu_1} = \eta + 2 \geq 2$.
Since $\eta = -2(\nu_1 - 2)\gamma_1$ where $\gamma_1 = -2\nu_1 + f + \nu_1 B$, we have the next two cases : case (1) $\eta = -2$ and case (2) $\eta = 0$ by $\sigma = 2\nu_1$.
In case (1), it follows that $2\nu_1 - 4 = 2$ and $\gamma_1 = 1$. Then $\sigma = 6, \nu_1 = 3$. Thus $f = 7 - 3B, g_0 = 30$. By genus formula,

$$t_2 + t_3 = 9, \quad t_2 + 3t_3 = 30 - 1 = 29.$$

Thus $2t_3 = 20$; hence, $t_2 < 0$; a contradiction.

In case (2), $\gamma_1 = 0$ and then $\zeta_{\nu_1} = 2$. Thus,

$$2 = \zeta_{\nu_1} = \sum_{j=3}^{\nu_1 - 1} (\nu_1 - j)(j-2)t_j.$$

Hence,

$$2 = (\nu_1 - 3)(t_3 + t_{\nu_1 - 1}) + 2(\nu_1 - 4)(t_4 + t_{\nu_1 - 2}) + \cdots .$$

Accordingly, we have the following two cases:

1. $\nu_1 - 3 = 1, t_3 = 2$,

2. $\nu_1 - 3 = 2, t_3 + t_4 = 1$.

In case (1), $\nu_1 = 4, \sigma = 8, g_0 = 49, t_3 = 2$; hence,

$$t_2 + t_3 + t_4 = 9, t_2 + 3t_3 + 6t_4 = 49 - 1.$$

Thus, $t_2 = 0, t_3 = 2, t_4 = 7, f = 8 - 4BB$ and the type is $[8 * 8; 4^7, 3^2]$ or its associates.

In case (2), $\nu_1 = 5, \sigma = 10, g_0 = 81, t_3 + t_4 = 1$; hence,

$$t_2 + t_3 + t_4 + t_5 = 9, t_2 + 3t_3 + 6t_4 + 10t_5 = 81 - 1.$$

Thus, $3t_5 + t_4 = 23$; a contradiction.

Combining these results, we establish the next result:

Theorem 4 *Suppose that $P_2[D] = 2g \geq 2$.*

1. *If $g = 3$, then $D^2 = 16$ and the type of the curve is $[4; 1]$.*

2. *If $g = 2$, then $D^2 = 4$ and the type is $[4 * 4; 2^7]$ or its associates.*

3. *If $g = 1$, then*

   (a) *if $D^2 = -2$, then the type is $[8 * 8; 4^7, 3^2]$ or its associates.*
   (b) *If $D^2 = -3$, then the type is $[6 * 6; 3^7, 2^3]$ or its associates.*
   (c) *If $D^2 = -4$, then the type is $[4 * 5; 2^{11}]$ or its associates.*

The pair defined by the curve $y^{10} = x^2(1-x)^3$ is birationally equivalent to a # minimal pair with type $[4 * 4; 2^7]$ where $g = 2$.

8 curves with $Z^2 = 2$

We shall study pairs (S, D) in the case when $Z^2 = 2$, i.e., $P_{2,1}[D] = Z^2 + 2 - g = 4 - g$. Thus it follows that $4 \geq g$.

8.1 case $g = 2, 3, 4$

If $g > 1$ then $P_2[D] = Z^2 + 2g - 1 = 1 + 2g$.

- If $g = 4$, then $P_2[D] = 9 = 3 \cdot 4 - 3$ and so by Theorem 3, $\sigma = 3$ and the type is $[3 * 3; 1]$ or $[3 * 6, 2; 1]$.

In the other cases, $g \leq 3$ and $\sigma \geq 4$. Moreover, $2 = Z^2 < Z_0^2$. Actually, $Z_0^2 = (\sigma - 2)(\sigma B + 2f - 4) \geq 4$. Hence, $\nu_1 \geq 2$.

- If $g = 3$, then $7 = P_2[D] = 3 \cdot 3 - 2$ and so by Theorem 7, the type is $[4 * 4; 2^6]$ or its associates.

- If $g = 2$, then $Z \cdot D = 2g - 2 = 2$ and $2 = Z^2 = Z \cdot D + Z \cdot K_S = 2 + Z \cdot K_S$. Hence, $Z \cdot K_S = 0$ and ${K_S}^2 = D^2 - 2, Q = (2Z - D)^2 = D^2$.

Claim 3 ${K_S}^2 < 0$.

Proof: Otherwise, ${K_S}^2 \geq 0$ and so $D^2 = {K_S}^2 + 2 \geq 2$. By Riemann-Roch, $\dim | - K_S| = {K_S}^2 \geq 0$. Hence, $(2Z - D) \cdot (-K_S) \geq 0$. From this, it follows that

$$0 \geq (2Z - D) \cdot K_S = (2Z - D) \cdot (Z - D) = 2Z^2 - 3Z \cdot D + D^2 = 4 - 6 + D^2.$$

Hence, $2 \geq D^2$. Therefore, $2 = D^2$. This implies that $(2Z - D) \cdot K_S = 0$. Noting that $Q = (2Z - D)^2 = D^2 = 2$,by Hodge's index theorem, we get $K_S \sim 0$ or ${K_S}^2 < 0$. But both cases cannot occur, because $K_S \not\sim 0$ and $K_S^2 \geq 0$ by hypothesis. $\square$

Since $\nu_1 \geq 2$, it follows that

$$0 \leq (\nu_1 Z - (\nu_1 - 1)D) \cdot (2Z - D) = 4 - 2\nu_1 + (\nu_1 - 1)D^2.$$

Hence,

$$D^2 \geq \frac{2\nu_1 - 4}{\nu_1 - 1} = 2 - \frac{2}{\nu_1 - 1}.$$

Suppose that $\nu_1 \geq 4$. Then $D^2 \geq 2$ and so ${K_S}^2 = D^2 - 2 \geq 0$. This is impossible due to the previous claim. Therefore, $\nu_1 = 2, 3$ and $D^2 = K_S^2 + 2 \leq 1$.

If $\nu_1 = 3$, then $D^2 \geq 2 - \dfrac{2}{\nu_1 - 1} = 1$. By Claim , $D^2 = K_S^2 + 2 \leq 1$;thus $D^2 = 1$. In this case, ${K_S}^2 = -1$ and $r = 9$. Furthermore, $A = Z^2 - \overline{g} = 1, \alpha = 4\overline{g} - D^2 = 3$. Hence, $(3Z - 2D)(2Z - D) = 6A - 2\alpha = 0$. But by $\nu_1 \leq 3$, $(3Z - 2D)(2Z - D) = (3Z_0 - 2C)(2Z_0 - C) = \tau_5 - 2$. Thus $\tau_5 = 2$ and so $\sigma = 6, \tilde{B} = 6B + 2f = 2\sigma = 12, g_0 = 25$. By genus formula,

$$t_2 + t_3 = r = 9; \quad t_2 + 3t_3 = g_0 - g = 23.$$

Immediately, we get $t_2 = 2, 3t_3 = 7$. Hence, the type is $[6 * 6; 3^7, 2^2]$ or its associates. [1]

If $\nu_1 = 2$, then

$$2 = 4 - 2 = (2Z - D) \cdot Z = \tau_3 - 2,$$

hence,

$$(\sigma - 3)(\tilde{B} - 6) = \tau_3 = 4.$$

Then $\sigma = 4, \tilde{B} = 10$. Therefore, the type is $[4 * 5; 2^{10}]$ or its associates.

8.2 case $g = 1$

If $g = 1$, then $P_2[D] = Z^2 + 2 = 4$ and $Z \cdot D = 2g - 2 = 0$ and ${K_S}^2 = 2 + D^2$. Thus

$$0 \leq (\nu_1 Z - (\nu_1 - 1)D) \cdot (2Z - D) = 4\nu_1 + (\nu_1 - 1)D^2.$$

Hence,

$$D^2 \geq \frac{-4\nu_1}{\nu_1 - 1} = -4 - \frac{4}{\nu_1 - 1}.$$

Claim 4

$$D^2 \leq -3.$$

Actually, if $D^2 \geq -2$ then ${K_S}^2 = 2 + D^2 \geq 0$. Hence, by Riemann-Roch, $|-K_S| \neq \emptyset$. Since $\sigma \geq 4$, it follows that

$$0 \geq (2Z - D) \cdot K_S = (2Z - D) \cdot (Z - D) = 2Z^2 + D^2 = 4 + D^2.$$

Hence, $-4 \geq D^2$. This contradicts the hypothesis. $\square$

[1] The author thanks S.Usuda who first noticed the existence of this case.

• Suppose that $\nu_1 \leq 2$. Then applying a corollary to Lemma 3 for $Z^2 = 2$, we obtain

$$(\sigma - 3)(2f + B\sigma - 6) = 6.$$

Thus letting $i = \sigma - 3$ be a divisor of 6, we obtain

$$B(i+3) + 2f - 6 = \frac{6}{i}$$

where $i = 1, 2$.

(1) If $B = 0$, then $2f - 6 = \frac{6}{i}$, which implies that $i = 1, \sigma = 4, f = 6$.

(2) If $B = 1$, then $i + 3 + 2f - 6 = \frac{6}{i}, f \geq 2$, which implies that $i = 1, 2$. Thus when $i = 1$, we get $\sigma = 4, f = 4$. While $i = 2$ induces $\sigma = 5, f = 2$.

(3) If $B \geq 2$, then $B(i+3) + 2f - 6 = \frac{6}{i} \geq 2(i+3) + 2f - 6$, which implies that $i = 1, B = 2, \sigma = 4, f = 2$.

Therefore, the type is $[5 * 7, 1; 2^{13}]$ or $[4 * 6; 2^{14}]$ or its associates. In the former case, $D^2 = -7$ and in the latter case $D^2 = -8$.

• Suppose that $\nu_1 \geq 3$. Then $D^2 \geq -4 - \dfrac{4}{\nu_1 - 1} \geq -6$.

Moreover, if $\nu_1 \geq 6$, then $D^2 \geq -4$. If $\nu_1 \geq 4$, then $D^2 \geq -5$.

In what follows we shall study pairs in the cases : $D^2 = -3, -4, -5, -6$.

8.3 case $D^2 = -3$

Suppose that $D^2 = -3$. Then ${K_S}^2 = D^2 + 2 = -1$ and so $r = 9$. Therefore, $\xi_0 = -1 + 3/2 = 1/2$ and $\alpha = 4 - 1 = 3$. By sharper estimate,

$$0 \leq \zeta_{\nu_1} = \tilde{\eta} + \nu_1 + 3.$$

If $p \geq 1$ then $\tilde{\eta} \leq (5 - 3\nu_1)p$; hence

$$\tilde{\eta} + (\nu_1 + 3)p \leq (1 - 3p)\nu_1 + 5p + 3.$$

Suppose that $\nu_1 \geq 4$. Then $p = 1, \nu_1 = 4$. Hence, $\dfrac{17}{2} - \dfrac{\sigma}{2} - f \leq 0$. Thus the equalities hold and then $\nu_1 = 4, \zeta_{\nu_1} = t_3 = 0, \sigma = 9, f = 4$; hence, $g_0 = 24 + 36 = 60$. By genus formula,

$$t_2 + t_3 + t_4 = 9, t_3 = 0, t_2 + 3t_3 + 6t_4 = 59.$$

Hence, $5t_4 = 50, t_4 = 10 > 9$; a contradiction.

Suppose that $\nu_1 = 3$. Then

$$(3Z-2D)(2Z-D) = (3Z_0-2C)(2Z_0-C) = \tau_5-2, (3Z-2D)(2Z-D) = 6Z^2+2D^2 = 6.$$

Hence, $\tau_5 = 8$. From

$$(\sigma - 5)(\tilde{B} - 10) = \tau_5 = 8$$

it follows that

$$\sigma - 5 = 2, \tilde{B} - 10 = 4.$$

Thus

$$\sigma = 7, \tilde{B} = 14; 7B + 2f = 14.$$

This implies that the type is $[7*7; 3^{t_3}, 2^{t_2}]$ or its associates. Moreover,$g_0 = 36$ and hence, by genus formula,

$$t_2 + t_3 = 9,$$

$$t_2 + 3t_3 = g_0 - 1 = 35.$$

Thus, $2t_3 = 26; t_3 = 13, t_2 = -4$, which is a contradiction.

Therefore, $p = 0$ has been established and so Formula II becomes

$$\zeta_{\nu_1} = \eta + \nu_1 + 3.$$

Supposing that $\eta \neq 0$, we shall derive a contradiction.

Recalling that $\eta = -2(\nu_1 - 2)\gamma_1$, we obtain

$$\zeta_{\nu_1} = \eta + \nu_1 + 3 = -2(\nu_1 - 2)\gamma_1 + \nu_1 + 3.$$

Assume that $\nu_1 \geq 4$. Then from $\zeta_{\nu_1} \geq 0$, it follows that $\gamma_1 = 1$. Hence, $\zeta_{\nu_1} = 7 - \nu_1$. Note that $\gamma_1 = 1$ implies that $\tilde{B} = 2(\nu_1 B + 2f) = 2 + 4\nu_1$. Hence, $g_0 = 2\nu_1(2\nu_1 - 1)$.

Moreover, note that

$$\zeta_{\nu_1} = F(\nu_1) = (\nu_1 - 3)x_1 + 2(\nu_1 - 4)x_2 + \cdots .$$

Thus if $\zeta_{\nu_1} \neq 0$, then $\zeta_{\nu_1} \geq \nu_1 - 3$, which implies that $\nu_1 \leq 5$.

• If $\nu_1 = 7$, then $\zeta_{\nu_1} = 0, \sigma = 14$ and $-10 = \eta = (\sigma - 4)(\sigma - f - \frac{B\sigma}{2})$. Thus $g_0 = 13 \cdot 14 = 182, t_3 = t_4 = t_5 = t_6 = 0$ and moreover,

$$r = t_2 + t_7 = 9, \quad t_2 + 21t_7 = g_0 - g = 182 - 1 = 181.$$

But from this, it follows that $10t_7 = 86$; a contradiction.

• If $\nu_1 = 5$, then $\sigma = 10$ and $\zeta_{\nu_1} = 7 - \nu_1 = 2$. By definition,

$$2 \geq \zeta_{\nu_1} = 2t_3 + 2t_4.$$

Hence, $t_3 + t_4 = 1$.

When $t_3 + t_4 = 1$, we get $2 = \zeta_{\nu_1} = \eta + \nu_1 + 3 = \eta + 8$; thus

$$-6 = \eta = (\sigma - 4)(\sigma - f - \frac{B\sigma}{2}) = 6(10 - f - 5B).$$

Hence, $11 = f + 5B$ and so $g_0 = 90$. Therefore,

$$t_3 + t_4 = 1, t_2 + t_3 + t_4 + t_5 = 9,$$

$$t_2 + 3t_3 + 6t_4 + 10t_5 = 90 - 1 = 89.$$

Hence, $t_4 + 3t_5 = 26$. But since $t_4 = 0$ or 1, it follows that $3t_5 = 26, 25$;a contradiction.

• If $\nu_1 = 4$, then $\sigma = 8$ and $t_3 = \zeta_{\nu_1} = 7 - \nu_1 = 3$. Then $g_0 = 2 \cdot 4 \cdot 7 = 56$. Hence,

$$t_3 \leq 3, t_2 + t_3 + t_4 = 9,$$

$$t_2 + 3t_3 + 6t_4 = 56 - 1 = 55.$$

Hence, $t_3 = 3, t_4 = 8, t_2 = -2$; a contradiction.

• If $\nu_1 = 3$, then

$(3Z - 2D)(2Z - D) = (3Z_0 - 2C)(2Z_0 - C) = \tau_5 - 2, (3Z - 2D)(2Z - D) = 6Z^2 + 2D^2 = 6.$

Hence, $\tau_5 = 8$. From

$$(\sigma - 5)(\tilde{B} - 10) = \tau_5 = 8$$

it follows that

$$\sigma - 5 = 1, \tilde{B} - 10 = 8.$$

Thus

$$\sigma = 6, \tilde{B} = 18; 6B + 2f = 18.$$

This implies that the type is $[6{*}9; 3^{t_3}, 2^{t_2}]$ or its associates. Moreover,$g_0 = 40$ and hence, by genus formula,

$$t_2 + t_3 = 9, \quad t_2 + 3t_3 = 40 - 1 = 39.$$

Hence, $t_3 = 15, t_2 = -6$; contradiction. Therefore, $\eta = 0$ is established. $\square$

8.3.1 case $\eta = 0$

$\eta = 0$ implies that if $\sigma > 4$, then $\sigma - f - \dfrac{B\sigma}{2} = 0$. In this case, $2g_0 = (\sigma-1)(2f+B\sigma-2) = 2(\sigma-1)^2$.

From the definition of ζ_{ν_1}, it follows that

$$\zeta_{\nu_1} = \nu_1 + 3 = (\nu_1-3)x_1 + 2(\nu_1-4)x_2 + 3(\nu_1-5)x_3 + \cdots ,$$

where

$$x_1 = t_3 + t_{\nu_1-1}, x_2 = t_4 + t_{\nu_1-2}, x_3 = t_5 + t_{\nu_1-3}, \cdots .$$

Define a function $F(n)$ to be $\sum_{p=1}^{\mu} p(n-p-2)x_p$ where $\mu = [\dfrac{n-2}{2}], x_p = t_{p+2} + t_{n-p}$.

Then the values of $F(n)$ are $n-3, 2(n-3), 2(n-4), 3(n-5), n-3+2(n-4), \cdots$.

Lemma 7 *If $p < q \leq \dfrac{n-2}{2}$, then $p(n-p-2) < q(n-q-2)$.*

Proof: $p(n-p-2) - q(n-q-2) = -(p-q)(p+q-(n-2)) < 0$. □

Hence, when $\zeta_{\nu_1} = \nu_1 + 3$, from $\nu_1 + 3 = F(\nu_1) \geq 2(\nu_1-4), 2(\nu_1-3)$, it follows that $\nu_1 \leq 11$. Thus, we shall study pairs in the following cases according to the value of $\nu_1 \leq 11$.

• If $\nu_1 = 11$, then $\sigma = 22, g_0 = 21^2 = 441$ and

$$\nu_1 + 3 = 14 = F(11) = 8x_1 + 14x_2 + 18x_3 + \cdots$$

;thus $x_1 = 0$, $x_2 = 1$. Since $t_3 = t_{10} = 0, t_4 + t_9 = 1, t_5 = t_8 = 0, t_6 = t_7 = 0$, it follows that

$$t_2 + t_4 + t_9 + t_{11} = 9, \quad t_2 + 6t_4 + 36t_9 + 55t_{11} = 440.$$

From this, we get

$$t_2 + t_{11} = 8, \quad 5t_4 + 35t_9 + 54t_{11} = 431, \quad 35t_9 + 54t_{11} = 431 - 5t_4.$$

Then $54t_{11} = 426$, or $54t_{11} = 396$; a contradiction.

• If $\nu_1 = 10$, then

$$13 = \nu_1 + 3 = F(10) = 7x_1 + 12x_2,$$

which is impossible.

• If $\nu_1 = 9$, then $\sigma = 18, g_0 = 17^2 = 289$ and

$$12 = F(9) = 6x_1 + 10x_2 + 12x_3.$$

Hence, here are two cases a)$x_1 = x_2 = 0, x_3 = 1$, and b) $x_1 = 2, x_2 = x_3 = 0$.

In case a), $t_3 = t_8 = 0, t_4 = t_7 = 0, t_5 + t_6 = 1$.

$$t_2 + t_5 + t_6 + t_9 = 9, \quad t_2 + 10t_5 + 15t_6 + 36t_9 = 288.$$

From these,

$$9t_5 + 14t_6 + 35t_9 = 279, \quad 5t_6 + 35t_9 = 270, \quad t_6 + 7t_9 = 54.$$

Since $t_6 = 0$ or 1, then $7t_9 = 54$ or 53; a contradiction.

In case b), $t_3 + t_8 = 2, t_4 = t_5 = t_6 = t_7 = 0$.

$$t_2 + t_3 + t_8 + t_9 = 9, \quad t_2 + 3t_3 + 28t_8 + 36t_9 = 288.$$

From these,

$$2t_3 + 27t_+ 35t_9 = 279, \quad 5t_8 + 7t_9 = 55.$$

Since $t_6 = 0$ or 1, 2, then $7t_9 = 55$ or 50,45; a contradiction.

• If $\nu_1 = 8$, then

$$11 = F(8) = 5x_1 + 8x_2 + 9x_3.$$

There exist no solutions.

• If $\nu_1 = 7$, then $\sigma = 14, g_0 = 13^2 = 169$ and

$$10 = F(7) = 4x_1 + 6x_2.$$

Then $x_1 = x_2 = 1$;thus $t_3 + t_6 = 1, t_4 + t_5 = 1$ and therefore,

$$t_2 + t_3 + t_4 + t_5 + t_6 + t_7 = 9, \quad t_2 + 3t_3 + 6t_4 + 10t_5 + 15t_6 + 21t_7 = 168.$$

Hence,

$$2+5+4t_5+12t_6+20t_7 = 159, \quad 4t_5+12t_6+20t_7 = 152, \quad t_5+3t_6+5t_7 = 38.$$

Then $t_7 = 7, t_2 = 0, t_4 = t_6 = 1, t_3 = t_5 = 0$. Thus

$$D^2 = 2 \cdot 14 \cdot 14 - 4 \cdot 4 - 6 \cdot 6 - 7 \cdot 7 \cdot 7 = -3.$$

The type is $[14 * 14; 7^7, 6, 4]$ or its associates.

• If $\nu_1 = 6$, then $\sigma = 12, g_0 = 11^2 = 121$ and

$$9 = F(6) = 3x_1 + 4x_2.$$

Thus, $x_1 = 3, x_2 = 0$, i.e. $t_4 = 0, t_3 + t_5 = 3$ and therefore,

$$t_2 + t_3 + t_5 + t_6 = 9, \quad t_2 + 3t_3 + 10t_5 + 15t_6 = 120.$$

Hence,

$$2t_3 + 9t_5 + 14t_6 = 111, \quad 7t_5 + 14t_6 = 105; \quad t_5 + 2t_6 = 15.$$

That is, $t_6 = 6, t_5 = 3, t_2 = t_3 = 0$ and so $D^2 = 2 \cdot 12^2 - 3 \cdot 5^2 - 6 \cdot 6^2 = -3$. The type is $[12 * 12; 6^6, 5^3]$ or its associates.

• If $\nu_1 = 5$, then $\sigma = 10, g_0 = 9^2 = 81$ and

$$8 = F(5) = 2x_1 = 2t_3 + 2t_4.$$

Then $x_1 = 4$, i.e. $t_3 + t_4 = 4$ and therefore,

$$t_2 + t_3 + t_4 + t_5 = 9, \quad t_2 + 3t_3 + 6t_4 + 10t_5 = 81 - 1 = 80,$$

$$12 + 5 + 3t_4 + 9t_5 = 80, \quad 3t_4 + 9t_5 = 63; \quad t_4 + 3t_5 = 21.$$

But $t_5 \leq 5, t_4 \leq 4$, which contradicts $t_4 + 3t_5 = 21$.

• If $\nu_1 = 4$, then $\sigma = 8, g_0 = 7^2 = 49$ and $7 = F(4) = t_3$. Moreover,

$$t_2 + t_3 + t_4 = 9, \quad t_2 + 3t_3 + 6t_4 = 49 - 1 = 48.$$

But $5t_4 = 25$; $t_4 = 5, t_3 = -1$; a contradiction.

• If $\nu_1 = 3$, then $\sigma = 6, g_0 = 5^2 = 25$ and by genus formula

$$t_2 + t_3 = 9, \quad t_2 + 3t_3 = 25 - 1 = 24.$$

But $2t_3 = 24 - 9 = 15$; a contradiction.

8.4 case $D^2 = -4$

Suppose that $D^2 = -4$. Then $K_S{}^2 = D^2 + 2 = -2$ and so $r = 10$. $\xi_0 = -1 + 1 = 0$ and $\alpha = 4$. Moreover, $\xi_2 = \sigma + f - 8 + \dfrac{B\sigma}{2}$, and

$$0 \le \zeta_{\nu_1} = \eta + 4 - \xi_2 p.$$

If $B \neq 1$, then $\eta \le 0$ and $0 \le \zeta_{\nu_1} \le 4 - \xi_2 p \le 4 - 6p$. Hence, $p = 0$.

If $B = 1$, then $\eta - \xi_2 p \le p(-2 + 10 - \sigma - f - 2) = p(6 - \sigma - f)$.

Supposing that $p > 0$, we get $\sigma \ge 7$ and $f \ge 3$. Hence, $0 \le \zeta_{\nu_1} = \eta + 4 - \xi_2 p \le 4 - 4p$, which implies that $p = 1, \sigma = 7, f = 3, g_0 = 12 + 21 = 33$. By genus formula, we get

$$t_2 + t_3 = 10, \quad t_2 + 3t_3 = 33 - 1 = 32.$$

Thus $2t_3 = 32 - 10 = 22; t_3 = 11 > 10$; a contradiction. Therefore, $p = 0$ is verified. By the formula, we get

$$0 \le \zeta_{\nu_1} = \eta + 4 \le 4.$$

Hence, $0 \le \eta + 4$.

8.4.1 case $\eta \neq 0$

If $\eta \neq 0$, then $\eta \le 4 - \sigma = 4 - 2\nu_1$.

Actually, $\eta = (\sigma - 4)(\sigma - f - B\sigma/2) = (2\nu_1 - 4)(2\nu_1 - f - B\nu_1) < 0$, that is a multiple of $2\nu_1 - 4$.

If $\zeta_{\nu_1} \ge 1$, then $\nu_1 \ge 4$ and so $\sigma \ge 2 \times 4 = 8$. Hence, when $\eta \neq 0$, $-\eta$ is a multiple of $2\nu_1 - 4 \ge 4$; thus $\eta = -4$ and $\zeta_{\nu_1} = 0$.

Therefore, we may assume that $\zeta_{\nu_1} = 0$ and then $\eta = -4$ and $-\eta = (2\nu_1 - 4)(2\nu_1 - f - B\nu_1)$. Hence,

$$4 = -\eta = -(2\nu_1 - 4)(2\nu_1 - f - B\nu_1).$$

Therefore, we have two cases (1) $\nu_1 - 2 = 2, 2\nu_1 - f - B\nu_1 = -1$, (2) $\nu_1 - 2 = 1, 2\nu_1 - f - B\nu_1 = -2$.

In case (1), $\nu_1 = 4, t_3 = 0, g_0 = 56$. By genus formula,

$$t_2 + t_4 = 10, \quad t_2 + 6t_4 = 56 - 1 = 55.$$

Then $t_2 = 1, t_4 = 9$. The type is $[8 * 9; 4^9, 2]$ or its associates.

In case (2), $\nu_1 = 3, g_0 = 35$. By genus formula,

$$t_2 + t_3 = 10, \quad t_2 + 3t_3 = 35 - 1 = 34.$$

Then $2t_3 = 24, t_3 = 12 > 10$; a contradiction.

8.4.2 case $\eta = 0$

Suppose that $\eta = 0$. Then $\zeta_{\nu_1} = 4$ and

$$4 = F(\nu_1) = (\nu_1 - 3)x_1 + 2(\nu_1 - 4)x_2 + \cdots .$$

From $4 \geq \nu_1 - 3$, it follows that $\nu_1 \leq 7$. Therefore, we examine in the following four cases:

• If $\nu_1 = 7$, then $\sigma = 14, g_0 = 13^2 = 169, 4 = F(7)$. Hence, $t_3 + t_6 = 1$ and $t_4 = t_5 = 0$. Thus,

$$t_2 + t_3 + t_6 + t_7 = 10, \quad t_2 + 3t_3 + 15t_6 + 21t_7 = 169 - 1 = 168.$$

Then

$$t_2 + t_7 = 9, \quad 2t_3 + 14t_6 + 20t_7 = 158,$$
$$t_3 + 7t_6 + 10t_7 = 79, \quad 3t_6 + 5t_7 = 39, t_6 \leq 1.$$

This is impossible.

• If $\nu_1 = 6$ then $\sigma = 12, g_0 = 11^2 = 121, \quad 4 = F(6) = 3x_1 + 4x_2$. Thus, $x_1 = 0$ and $x_2 = 1$;hence, $t_3 = t_5 = 0, t_4 = 1$ and so

$$t_2 + t_4 + t_6 = 10, \quad t_2 + 6t_4 + 15t_6 = 121 - 1 = 120.$$

Therefore,

$$t_2 + t_6 = 9, \quad t_2 + 15t_6 = 120 - 6 = 114.$$

Thus $14t_6 = 114 - 9 = 105$; a contradiction.

• If $\nu_1 = 5$ then $\sigma = 10, g_0 = 9^2 = 81, 4 = 2x_1$. Thus, $x_1 = 2$;hence, $t_3 + t_4 = 2$.

$$t_2 + t_3 + t_4 + t_5 = 10, \quad t_2 + 3t_3 + 6t_4 + 10t_5 = 80.$$

Hence,

$$t_2 + t_5 = 8, \quad 2t_3 + 5t_4 + 9t_5 = 70,$$
$$t_4 + 3t_5 = 22.$$

Finally, $t_2 = t_3 = t_4 = 1, t_5 = 7$. Thus the type is $[10 * 10; 5^7, 4, 3, 2]$ or its associates.

• If $\nu_1 = 4$, then $\sigma = 8, g_0 = 7^2 = 49, \zeta_{\nu_1} = 4$ and $\zeta_{\nu_1} = t_3$, i.e. $t_3 = 4$. Hence,

$$t_2 + t_3 + t_4 = 10, \quad t_2 + 3t_3 + 6t_4 = 48.$$

Hence, $t_2 = 0, t_3 = 4, t_4 = 6$. The type is $[8 * 8; 4^6, 3^4]$ or its associates.

8.5 case $D^2 = -5$

Suppose that $D^2 = -5$. Then ${K_S}^2 = D^2 + 2 = -3$ and so $r = 11$. $\xi_0 = 5 + 5/2 + 1 - 11 = -1/2$ and $\alpha = 5$. Moreover, $\xi_2 = \sigma + f + \frac{B\sigma}{2} - \frac{17}{2}$ and

$$0 \le \zeta_{\nu_1} = \eta + 5 - \frac{\sigma}{2} - \xi_2 p.$$

Suppose that $p \ge 1, \nu_1 \ge 3$. Then $\sigma \ge 7$; hence, $\xi_2 \ge 5$.

If $B \ne 1$ then $0 \le \zeta_{\nu_1} \le 5 - \frac{\sigma}{2} - 5p < 0$; a contradiction.

If $B = 1$ then $\eta \le \frac{\sigma - 4}{2} p$ and so

$$0 \le \zeta_{\nu_1} \le 5 - \frac{\sigma}{2} - (\xi_2 - \frac{\sigma - 4}{2})p.$$

But, $\xi_2 - \frac{\sigma - 4}{2} = \sigma + f - \frac{13}{2} - 6 \ge 0$ and

$$5 - \frac{\sigma}{2} - (\xi_2 - \frac{\sigma - 4}{2})p \le 5 - \frac{\sigma}{2} - \sigma + f - \frac{13}{2} \le -2.$$

This implies that $p = 0$. In particular, $\eta = \sigma - f - \frac{B\sigma}{2} = \frac{\sigma}{2} - f = \nu_1 - f \le 0$. Therefore, in both cases, $\eta \le 0$ and hence,

$$0 \le \zeta_{\nu_1} = \eta - \nu_1 + 5 \le -\nu_1 + 5.$$

• If $\nu_1 = 5$, then $\zeta_{\nu_1} = 0, \sigma = 10, \eta = 0$. Hence, $t_3 = t_4 = 0, g_0 = 81$. By genus formula

$$t_2 + t_5 = 11, \quad t_2 + 10t_5 = 81 - 1 = 80.$$

Hence, $9t_5 = 80 - 11 = 69$; a contradiction.

• If $\nu_1 = 4$, then $\sigma = 8, \eta = (\sigma - 4)(\sigma - f - B\sigma/2) = 4(8 - f - 4B) = -1$ or 0. Hence, $\eta = 0$ and thus $\zeta_{\nu_1} = t_3 = 1, g_0 = 49$. By genus formula,

$$t_2 + t_3 + t_4 = 11, \quad t_2 + 3t_3 + 6t_4 = 49 - 1 = 48.$$

Hence, $5t_4 = 35, t_4 = 7, t_2 = 3$. The type is $[8 * 8; 4^7, 3, 2^3]$ or its associates.

• If $\nu_1 = 3$ then $\sigma = 6, \zeta_{\nu_1} = 0, \eta = -2$. Hence, $f + 3B = 7; g_0 = 30$. By genus formula,

$$t_2 + t_3 = 11, \quad t_2 + 3t_3 = 29.$$

Hence, $t_3 = 9$ and $t_2 = 2$. The type is $[6 * 7; 3^9, 2^2]$ or its associates.

8.6 case $D^2 = -6$

Suppose that $D^2 = -6$. Then $\nu_1 = 3, {K_S}^2 = D^2 + 2 = -4$ and so $r = 12$. By the same argument as before, $p = 0, \sigma = 6$ are obtained and so $g_0 = 5(3B + f - 1)$. By genus formula,

$$t_2 + t_3 = 12, \quad t_2 + 3t_3 = g_0 - 1.$$

Hence, $24 \geq 2t_3 = g_0 - 13 = 15B + 5f - 18$, which implies that $t_3 = \frac{15B + 5f}{2} - 9 \leq 12$. Therefore, $3B + f \leq 8$ and thus the type is $[6 * 6; 3^6, 2^6]$ or its associates.

Theorem 5 *Suppose that $Z^2 = 2$. if $g > 1$, $P_2[D] = Z^2 + 2g - 1 = 2g + 1$.*

1. *If $g = 4$, then $D^2 = 18$ and the type is $[3 * 3; 1]$ or $[3 * 6, 2; 1]$.*
2. *If $g = 3$, then $D^2 = 8$ and the type is $[4 * 4; 2^6]$ or its associates.*
3. *If $g = 2$, then either (1) $D^2 = 1$ and the type is $[6 * 6; 3^7, 2]$ or (2) $D^2 = 0$ and the type is $[4 * 5; 2^{10}]$ or its associates.*
4. *If $g = 1$ then $P_2[D] = Z^2 + 2 = 4$.*
   - (a) *If $D^2 = -3$, then the type is $[14 * 14; 7^7, 6^4]$ or $[12 * 12; 6^6, 5^3]$ or its associates.*
   - (b) *If $D^2 = -4$, then the type is $[8 * 8; 4^6, 3^4]$ or $[8 * 9; 4^9, 2]$ or $[10 * 10; 5^7, 4, 3, 2]$ or their associates.*
   - (c) *If $D^2 = -5$, then the type is $[6 * 7; 3^9, 2^2]$ or $[8 * 8; 4^7, 3, 2^3]$ or their associates.*
   - (d) *If $D^2 = -6$, then the type is $[6 * 6; 3^6, 2^6]$ or its associates.*
   - (e) *If $D^2 = -7$, then the type is $[5 * 7, 1; 2^{13}]$.*
   - (f) *If $D^2 = -8$, then the type is $[4 * 6; 2^{14}]$ or its associates.*

9 curves with $Z^2 = 3$

Assume that $P_2[D] = 3g$. Then $Z^2 = g + 1$ and hence, $g + 1 = Z^2 = K_S^2 - D^2 + 4g - 4$. First , if the type is $[d; 1]$ then $d = 7, g = 15, Z^2 = 16$. Second, assume that (S, D) is derived from a # minimal model.

Defining l to be $4g - D^2$, we obtain $D^2 = 4g - l$ and $K_S^2 = 5 + g - l$. From $K_S^2 = 8 - r$, it follows that $r = 3 + l - g$. Then by definition,

$$\xi_0 = 4 - \frac{l}{2}, \alpha = l - 4, \xi_2 = \sigma + f - 8 + \frac{B\sigma}{2} + 4 - \frac{l}{2}$$

We shall give an estimate of the magnitude of l.

Lemma 8 *If* $5 + g \geq l$ *then* $l \geq 8$.

Proof: By $K_S^2 = 5 + g - l \geq 0$, we have $|D - Z| = |-K_S| \neq \emptyset$. Hence, $(2Z - D) \cdot (Z - D) \leq 0$. Therefore,

$$2Z^2 - 3Z \cdot D + D^2 \leq 0.$$

Hence,

$$2(g + 1) = 2Z^2 \leq 6\overline{g} - D^2,$$

and so $8 \leq l$. □

If $l \leq 6$ then applying the previous lemma, we get $5 + g \geq 6 \geq l$ and thus, $l \geq 8$; a contradiction.

If $l = 7$ and $g \geq 2$ then $5 + g \geq 7 = l$ and hence, $l \geq 8$; a contradiction.

Therefore, in the case when $l = 7$, we may assume that $g = 1$. Then $Z^2 = 2$ and $D^2 = -7$. By Theorem 8, the type is $[5 * 7, 1; 2^{13}]$.

When $l \geq 8$, we shall consider in the following two cases: A) case $\nu_1 \geq 3$ and B) case $\nu_1 \leq 2$.

9.0.1 A) case $\nu_1 \geq 3$

In order to study the case when $l \geq 8$, we begin with the case in which $\sigma \geq 6$. Then $|3Z - 2D| \neq \emptyset$ by Theorem 1 and since $2Z - D$ is nef, it follows that

$$(3Z - 2D) \cdot (2Z - D) \geq 0,$$

and hence,

$$6Z^2 - 7Z \cdot D + 2D^2 \geq 0.$$

By

$$6Z^2 - 7Z \cdot D + 2D^2 = 6(g + 1) - 14\overline{g} + 2(4g - l) = 20 - 2l,$$

we obtain $l \leq 10$;hence, $l = 8, 9, 10$.

Moreover,

$$\begin{aligned} 0 \leq \zeta_{\nu_1} =& \eta + \xi_0\sigma + \alpha - \xi_2 p \\ =& \eta + (4 - \frac{l}{2})\sigma + l - 4 - \xi_2 p. \end{aligned}$$

To show that $p = 0$, we assume $p \geq 1$. Then $\sigma = p + 2\nu_1 \geq 7$ and since $l \geq 8$, it follows that

$$\begin{aligned} 0 \leq \zeta_{\nu_1} <& \eta + 4\sigma + 4 - \frac{l\sigma - l}{2} + l - \sigma - f - \frac{B\sigma}{2} \\ =& \eta + 3\sigma - f + \frac{3l - \sigma l}{2} - \frac{B\sigma}{2}. \end{aligned}$$

First assume $B \neq 1$. Then by $\sigma \geq 7$,we get

$$\begin{aligned} 0 \leq \zeta_{\nu_1} <& (3 - \frac{l}{2} - \frac{B}{2})\sigma + \frac{3l}{2} - f \\ \leq& 21 - 2l - \frac{7B}{2} - f. \end{aligned}$$

However, since $l \geq 8$, it follows that

$$21 - 2l - \frac{7B}{2} - f \leq 5 - \frac{7B}{2} - f \leq -2.$$

Second, assume that $B = 1$. Then recalling that $\sigma \geq 7, f \geq 3, B = 1$, we get

$$\begin{aligned} 0 \leq \zeta_{\nu_1} \leq& (4 - \frac{l}{2})\sigma + (l - 4) + \eta - \xi_2 p \\ \leq& (4 - \frac{l}{2})\sigma + (l - 4) + (2 + \frac{l}{2} - f - \sigma)p \\ \leq& (4 - \frac{l}{2})\sigma + (l - 4) + (2 + \frac{l}{2} - f - \sigma) \\ =& \frac{3l}{2} - 2 - f + (3 - \frac{l}{2})\sigma \\ \leq& \frac{3l}{2} + 21 - 2 - f - \frac{7l}{2})\sigma \\ \leq& 19 - f - 2l \leq 0. \end{aligned}$$

Hence,

$$0 \leq \zeta_{\nu_1} \leq 19 - f - 2l \leq 0.$$

If $\zeta_{\nu_1} = 0$,then $l = 8, \sigma = 7, p = 1, \nu_1 = 3, \tilde{B} = 13, g_0 = 33$. By genus formula,

$$t_2 + t_3 = r = 11 - g, \quad t_2 + 3t_3 = r = g_0 - g = 33 - g.$$

Hence,$2t_2 = -2g$. This implies that $g = 0$.

This is a contradiction and thus $p = \sigma - 2\nu_1 = 0$ is checked. Therefore,

$$\zeta_{\nu_1} = \eta + (4 - \frac{l}{2})\sigma + l - 4$$

has been established.

9.1 case $D^2 = 4g - 8$

Then $l = 8, r = 3 + l - g = 11 - g$. If $g = 1$ then $Z^2 = 2, D^2 = 4g - 8 = -4$. This case has been already treated in Theorem 5. So we may assume $g \geq 2$. Since $\sigma = 2\nu_1$ we get

$$\zeta_{\nu_1} = \eta + 4.$$

9.1.1 case $\eta = 0$

If $\eta = 0$ then $\sigma - f - B\sigma/2 = 0$ and $\zeta_{\nu_1} = 4$; hence, we obtain the equation:

$$4 = F(\nu_1) = (\nu_1 - 3)x_1 + 2(\nu_1 - 4)x_2 + \cdots.$$

Then from $4 \geq \nu_1 - 3$, it follows that $\nu_1 \leq 7$.

- If $\nu_1 = 7$ then $x_1 = 1$ and hence, $\sigma = 14, g_0 = 13^2 = 169$ and $t_3 + t_6 = x_1 = 1, t_4 = t_5 = 0$, which yields

$$t_2 + t_3 + t_6 + t_7 = 11 - g, t_2 + 3t_3 + 15t_6 + 21t_7 = 169 - g.$$

Thus

$$t_2 + t_7 = 10 - g, 2t_3 + 14t_6 + 20t_7 = 158; 6t_6 + 10t_7 = 78, \quad 3t_6 + 5t_7 = 39, t_6 \leq 1.$$

This is impossible.

- If $\nu_1 = 6$, then $\sigma = 12, g_0 = 11^2 = 121$ and $4 = F(6) = 3x_1 + 4x_2$. Thus, $x_1 = 0, x_2 = 1$;hence, $t_3 = t_5 = 0, t_4 = 1$ and so

$$t_2 + t_4 + t_6 = 11 - g, \quad t_2 + 6t_4 + 15t_6 = 121 - g.$$

Accordingly,

$$5t_4 + 14t_6 = 110.$$

Thus $14t_6 = 110 - 5t_4 = 105$; a contradiction.

• If $\nu_1 = 5$, then $\sigma = 10, g_0 = 9^2 = 81, 4 = F(5) = 2x_1$. Thus, $x_1 = 2$;hence, $t_3 + t_4 = 2$ and therefore,

$$t_2 + t_3 + t_4 + t_5 = 11 - g, \quad t_2 + 3t_3 + 6t_4 + 10t_5 = 81 - g.$$

Hence,

$$2t_3 + 5t_4 + 9t_5 = 70, \quad t_4 + 3t_5 = 22.$$

Then $t_4 = 1, t_3 = 1, t_5 = 7, t_2 = \varepsilon$; thus $g = 2 - \varepsilon$ and the type is $[10 * 10; 5^7, 4, 3, 2^\varepsilon]$ or its associates.

• If $\nu_1 = 4$, then $\sigma = 8, g_0 = 7^2 = 49, \zeta_{\nu_1} = 4$ and $\zeta_{\nu_1} = t_3$, i.e. $t_3 = 4$. Hence,

$$t_2 + t_3 + t_4 = 11 - g, \quad t_2 + 3t_3 + 6t_4 = 49 - g.$$

Hence, $2t_3 + 5t_4 = 38; t_4 = 6, t_2 = 1 - g$. Thus $g = 1$ and the type is $[8 * 8; 4^6, 3^4]$ or its associates.

9.1.2 case $\eta \neq 0$

If $\eta \neq 0$ then $\eta = -4, \nu_1 = 3, 4; \zeta_{\nu_1} = 0$ and therefore, we have two cases: $(1)\nu_1 = 3$ and (2) $\nu_1 = 4$.

• If $\nu_1 = 3$, then from $\eta = -4, \eta = 2(\nu_1 - 2)(4\nu_1 - 2\nu_1 B - f)$, it follows that $3B + f = 8$ and so $g_0 = 35$. By genus formula

$$t_2 + t_3 = 11 - g, \quad t_2 + 3t_3 = 35 - g.$$

Hence, $2t_3 = 24, t_3 = 12, t_2 < 0$; a contradiction.

• If $\nu_1 = 4$, then by the same argument as before, $\zeta_4 = 0, t_3 = 0, (2 - B)4 - f = -1$ and $g_0 = 56$. By genus formula

$$t_2 + t_4 = 11 - g, \quad t_2 + 6t_4 = 56 - g.$$

Hence, $5t_4 = 45; t_4 = 9$. And hence $t_2 = 2-g$ and the type is $[8*9; 4^9, 2^\varepsilon], g = 2 - \varepsilon$ or its associates, where $D^2 = 4g - 8$.

9.2 case $D^2 = 4g - 9$

Suppose that $D^2 = 4g - 9$. Then $l = 9$ and $r = 12 - g$.
Therefore,

$$0 \leq \zeta_{\nu_1} = \eta - \nu_1 + 5 \leq -\nu_1 + 5.$$

• case $\nu_1 = 5$ Then $\zeta_{\nu_1} = 0, \sigma = 10$ and $\eta = 0$. Hence, $t_3 = t_4 = 0$ and $g_0 = 81$. By genus formula

$$t_2 + t_5 = 12 - g, \quad t_2 + 10t_5 = 81 - g.$$

Hence, $9t_5 = 81 - 12 = 69$; a contradiction.

• case $\nu_1 = 4$ Then $\sigma = 8, 0 \leq \zeta_{\nu_1} = \eta + 1; -1 \leq \eta$. Moreover,

$$\eta = (\sigma - 4)(\sigma - f - B\sigma/2) = 4(8 - f - 4B) = -4, 0.$$

Hence, $\eta = 0$ and thus $\zeta_{\nu_1} = t_3 = 1, g_0 = 49$. By genus formula,

$$t_2 + t_3 + t_4 = 12 - g, \quad t_2 + 3t_3 + 6t_4 = 49 - g.$$

Hence, $5t_4 = 35, t_4 = 7$ and $t_2 = 4 - g$. The type is $[8 * 8; 4^7, 3, 2^{4-g}]$ or its associates, where $g = 1, 2, 3, 4$.

• case $\nu_1 = 3$. Then $\zeta_{\nu_1} = 0$ and $\sigma = 6, \zeta_{\nu_1} = \eta + 2$; thus $\eta = -2$. Hence, $f + 3B = 7$ and $g_0 = 30$. By genus formula,

$$t_2 + t_3 = 12 - g, \quad t_2 + 3t_3 = 30 - g.$$

Hence, $t_3 = 9, t_2 = 3 - g$. The type is $[6 * 7; 3^9, 2^{3-g}]$ or its associates where $g = 1, 2, 3$.

9.3 case $D^2 = 4g - 10$

Suppose that $D^2 = 4g - 10$. Then $l = 10, r = 13 - g$,

$$0 \leq \zeta_{\nu_1} = \eta - 2\nu_1 + 6 \leq -2(\nu_1 - 3).$$

Hence, $\nu_1 = 3, \sigma = 6$ and $\eta = 0, 3B + f = 6$. Clearly, $g_0 = 25$.
By genus formula,

$$t_2 + t_3 = 13 - g, \quad t_2 + 3t_3 = 25 - g.$$

Hence, $t_3 = 6$ and $t_2 = 7 - g$. The type is $[6 * 6; 3^6, 2^{7-g}]$ or its associates, where $g = 1, 2, \cdots, 7$.

9.3.1 B) case $\nu_1 \leq 2$

Since $\nu_1 \leq 2$, it follows that

$$4 = 2(g+1) - 2\overline{g} = 2Z^2 - D \cdot Z = (2Z - D) \cdot Z = \tau_3 - 2.$$

Hence, $\tau_3 = 6$. From

$$(\sigma - 3)(2f + B\sigma - 6) = 6,$$

we obtain either (1) $\sigma - 3 = 1, 2f + B\sigma - 6 = 6$ or (2) $\sigma - 3 = 2, 2f + B\sigma - 6 = 3$.

case (1) $\sigma = 4, 2f + B\sigma = 12, g_0 = 15$. The type is $[4 * 6; 2^r]$ and its associates , where $g = 15 - r = 1, 2, \cdots, 14$ and $D^2 = 4g - 12$.

case (2) $\sigma = 5, 2f + B\sigma = 9, g_0 = 14$ and the type is $[5 * 7, 1; 2^r]$, where $g = 14 - r$ and $D^2 = 4g - 11$.

Accordingly, we establish the following result:

Theorem 6 *Suppose that $P_2[D] = 3g > 1$. Then $Z^2 = g + 1$ and*
- *case $S = \mathbf{P}^2$. Then the type is $[7; 1]$ and $g = 15, D^2 = 49$.*
- *case $\nu_1 \leq 2$. Then the type is (1) $[4 * 6; 2^r]$ or its associates , where $g = 15 - r$ and $D^2 = 4g - 12$, or (2) $[5 * 7, 1; 2^r]$, where $g = 14 - r$ and $D^2 = 4g - 11$.*
- *case $\nu_1 \geq 3$. Then*

1. *if $5 \geq g \geq 7$ then the type is $[6 * 6; 3^6, 2^{7-g}]$ or its associates , where $D^2 = 4g - 10$.*
2. *If $g = 4$ then*
   (a) *if $D^2 = 7$ then the type is $[8 * 8; 4^7, 3]$ or its associates.*
   (b) *If $D^2 = 6$ then the type is $[6 * 6; 3^6, 2^3]$ or its associates.*
3. *If $g = 3$ then*
   (a) *if $D^2 = 2$ then the type is $[6 * 6; 3^6, 2^4]$ or its associates.*
   (b) *If $D^2 = 3$ then the type is $[8 * 8; 4^7, 3, 2]$ or $[6 * 7; 3^9]$ or their associates.*
4. *If $g = 2$ then*
   (a) *if $D^2 = 0$ then the type is either $[10 * 10; 5^7, 4, 3]$ or $[8 * 9; 4^9]$ or their associates.*

(b) *If* $D^2 = -1$ *then the type is either* $[6 * 7; 3^9, 2]$ *or* $[8 * 8; 4^7, 3, 2^2]$ *or their associates.*

(c) *If* $D^2 = -2$ *then the type is* $[6 * 6; 3^6, 2^5]$ *or its associates.*

5. *If* $g = 1$ *then*

(a) *if* $D^2 = -4$ *then the type is* $[8 * 8; 4^6, 3^4]$ *or* $[8 * 9; 4^9, 2]$ *or* $[10 * 10; 5^7, 4, 3, 2]$ *or their associates.*

(b) *If* $D^2 = -5$ *then the type is* $[6 * 7; 3^9, 2^2]$ *or* $[8 * 8; 4^7, 3, 2^3]$ *or their associates.*

(c) *If* $D^2 = -6$ *then the type is* $[6 * 6; 3^6, 2^6]$ *or its associates.*

When $\sigma = 3$, the invariants are easily computed:

$$A = Z^2 - \overline{g} = -1, \alpha = \overline{g} - 9, \omega = -9, \Omega = -3 - \overline{g}.$$

Moreover if the type is $[d; 1], d \geq 4$, then

$$A = \frac{(d-3)(d-6)}{2}, \alpha = d(d-6), \omega = \frac{d(d-9)}{2}, \Omega = (d-3)(d-9).$$

10 curves with $P_{2,1}[D] = 1$

By Lemma 3, when $\sigma \geq 4$, we see that $2Z - D$ is nef and so $(2Z - D) \cdot Z \geq 0$. Hence, $2Z^2 \geq D \cdot Z = 2\overline{g}$, i.e. $Z^2 \geq \overline{g}$. Thus we shall study pairs (S, D) with $Z^2 = \overline{g}$. Hence, $(2Z - D) \cdot Z = 0$ and $P_{2,1}[D] = Z^2 - \overline{g} + 1 = 1$; $Q = 0$. Noting that $\sigma \geq 4$ or $d \geq 6$ for the type $[d; 1]$, by Lemma 3, we get $\sigma = 4$ and $2Z - D \sim 0$ or $d = 6$.

Thus

$$0 \sim 2Z - D = D + 2K_S \sim C + 2K_0 \sim (f - 4 + 2B)F_c.$$

Hence, $f - 4 + 2B = 0$. Therefore, the type of the curve turns out to be $[4 * 4; 2^r]$ or its associates where $r \leq 7$. The pair is birationally equivalent to a pair of type $[6; 2^{r+1}]$. Thus we obtain the following result.

Theorem 7 *If* $P_{2,1}[D] = 1$, *then* $A = 0, Z^2 = \overline{g}$ *and the type is* $[6; 1]$ *or* $[4 * 4; 2^r]$ *or its associates where* $r \leq 7$.

Corollary 4 *Under the assumption $\sigma \geq 4$, $P_{2,1}[D] = A + 1 = 1$ if and only if $2Z \sim D$, i.e.$D + 2K_S \sim 0$.*

Proof:

From the formula $P_{2,1}[D] = Z^2 - g + 2$, the result follows immediately.

□

Definition 4 *If the pair (S, D) satisfies that $D + mK_S \sim 0$, then D is said to be an anti $m-$ canonical curve .*

The pair defined by $y^{3m} = x^m \prod_{j=1}^{m}(x - j)$ has the minimal model (S, D) of type $[2m * 3m, 1; m^5]$, which is an anti $m-$ canonical curve for $m > 1$.

So the theorem states that if $P_2[D] = 3g - 2 > 0$, then the curve D is anti-bicanonical.

11 curves with $P_{2,1}[D] = 2$

Suppose that $P_{2,1}[D] = 2$. Then $Z^2 = g$ and $Z \cdot D = 2g - 2$.

First, consider the case in which $\nu_1 \leq 2$.

Lemma 9 *If $Z^2 = g + i$ where the type is $[d; 1]$, then $(d-3)(d-6) = 2i + 2$.*

Proof: By $Z^2 = (d - 3)^2, Z \cdot D = d(d - 3) = 2g - 2$, we obtain $d(d - 3) = 2g - 2 = 2(d - 3)^2 - 2i - 2$ and then $(d - 3)(d - 6) = 2i + 2$. □

In the case when $i = 0$, there exists no solutions. Thus we consider the case where (S, D) is derived from a # minimal pair (σ_B, C). Applying Corollary 2 to the case $Z^2 = g$, we obtain

$$\tau = (\sigma - 3)(\tilde{B} - 6) = 4.$$

Hence, $\sigma - 3$ takes one of the following values 1, 2.

(1) If $\sigma = 5$, then $5B + 2f = 8$, which is impossible.

(2) If $\sigma = 4$, then $4B + 2f = 10$. Then (f, B)= (5, 0), (3, 1), (1, 2). $g_0 = 12 \geq r \geq 1$. Thus the type is $[4 * 5; 2^r]$, where $g = 12 - r$.

Second, consider the case in which $\nu_1 \geq 3$. By $|D + 3K_S| \neq \emptyset$, we get $(3Z-2D)\cdot Z = (D+3K_S)\cdot Z \geq 0$ and $(3Z-2D)\cdot Z = 3Z^2-2Z\cdot D = 3g-4g+4$; thus $g \leq 4$.

• Suppose that $g = 4$. Then $(3Z - 2D) \cdot Z = 0$. Since Z is nef and big, by Hodge's index theorem , we get $(3Z - 2D)^2 < 0$ or $3Z \sim 2D$. However,

$$0 \geq (3Z - 2D) \cdot (3Z - 2D) = (3Z - 2D) \cdot (-2D),$$

$$0 \leq (3Z - 2D) \cdot (2Z - D) = (3Z - 2D) \cdot (-D).$$

Hence, $(3Z - 2D) \cdot (-D) = 0$. Thus, $3Z \sim 2D$, i.e. $D \sim -3K_S$. Since $D + \nu_1 K_S \sim (\nu_1 - 3)K_S$ and $\kappa(S, K_S) = -\infty$, it follows that $\nu_1 = 3$. Therefore,

$$0 \sim D + 3K_S \sim C + 3K_0 + \sum_{j=1}^{r} (3 - \nu_j)E_j.$$

Hence, $\nu_1 = \cdots = \nu_r = 3$ and

$$0 \sim C + 3K_0 \sim (\sigma - 6)\Delta_\infty + (e - 6 - 3B)F_c.$$

Thus $\sigma = 6, e - 6 - 3B = 0$;i.e. $e = 6 + 3B$. Therefore, $g_0 = 25, 25 - 3t_3 = 4$; hence, $t_3 = 7$. This implies that the type is $[6 * 6; 3^7]$ or its associates.

• Suppose that $g = 3$. Then $D \cdot Z = 4, (3Z - 2D) \cdot Z = 1, (3Z - 2D) \cdot D = 2(6 - D^2)$. Since $2Z - D$ is nef, it follows that

$$(3Z - 2D) \cdot (2Z - D) \geq 0, \quad (3Z - 2D) \cdot (2Z - D) = 2 - 12 + 2D^2.$$

Hence, $D^2 \geq 5$.

On the other hand, $3 = Z^2 = {K_S}^2 - D^2 + 8$; thus ${K_S}^2 = D^2 - 5 \geq 0$. Hence, by Riemann-Roch, we get $|-K_S| \neq \emptyset$ and so

$$0 \leq (-K_S) \cdot (2Z - D) = (D - Z) \cdot (2Z - D) = 6 - D^2.$$

Thus $D^2 \leq 6$. Combining the previous results with $D^2 \geq 5$, we obtain $D^2 = 5, 6$.

11.1 case $D^2 = 6$

Then ${K_S}^2 = 1, r = 7$; thus $\xi_0 = 7 - 3 + 3 - 7 = 0, \quad \alpha = 8 - 6 = 2$, and $\xi_2 = \sigma + f - 1 + \dfrac{B\sigma}{2} + g - \dfrac{D^2}{2} - r = \sigma + f + \dfrac{B\sigma}{2} - 8$.

We shall verify that $p = 0$. Actually, suppose that $p \geq 1$.

$\sigma + f - 8 + \dfrac{B\sigma}{2} \geq 8$, provided

If $B \neq 1$ then $\xi_2 \geq 6, 0 \leq \zeta_{\nu_1} \leq 2 - 6p < 0$; a contradiction.

If $B = 1$ then $\eta - \xi_2 p \leq (3 + 7 - 4 - f - \sigma)p \leq -4p$. Thus $0 \leq \zeta_{\nu_1} \leq 2 - 4p < 0$; a contradiction.

Therefore, $p = 0$ and so by the formula,

$$0 \leq \zeta_{\nu_1} = \eta + 2 \leq 2.$$

If $\eta = 0$ then

$$2 = (\nu_1 - 3)x_1 + 2(\nu_1 - 4)x_2 + \cdots.$$

We have the following cases:

1) $\nu_1 - 3 = 2, x_1 = t_4 + t_3 = 1$. Then $\nu_1 = 5, \sigma = 10$ and $g_0 = 81$. Moreover,

$$t_2 + t_3 + t_4 + t_5 = 7, \quad t_2 + 3t_3 + 6t_4 + 10t_5 = 81 - 3 = 78.$$

Thus $t_2 + t_5 = 6, 3t_4 + 9t_5 = 69$;hence, $t_4 + 3t_5 = 23$. A contradiction.

2) $\nu_1 - 3 = 1, t_3 = 2$. Then $\nu_1 = 4, \sigma = 8$ and $g_0 = 49$. Further,

$$t_2 + t_3 + t_4 = 7, \quad t_2 + 3t_3 + 6t_4 = 49 - 3 = 46.$$

Thus $t_2 + t_4 = 5, 2t_3 + 5t_4 = 39$;hence, $5t_4 = 39 - 4 = 35, t_4 = 7$; a contradiction.

If $\eta < 0$ then by $\nu_1 \geq 3$, we see that $\sigma - 4 \geq 2$ and so $\eta = -2, \zeta_{\nu_1} = 0$. Then $6 - f - 3B = -1$. Hence, $g_0 = 30$. But $t_2 + t_3 = 7$ and $t_2 + 3t_3 = 30 - 3 = 27$; a contradiction.

11.2 case $D^2 = 5$

Then ${K_S}^2 = 0$ and $r = 8$; thus $\xi_0 = 7 - 5/2 + 3 - 8 = -1/2, \quad \alpha = 8 - 5 = 3, \xi_2 = \sigma + f - 8 + \frac{B\sigma}{2} - \frac{1}{2}$.

Suppose that $p \geq 1$.

Since $\nu_1 \geq 3$, it follows that $\xi_2 \geq \frac{3}{2}$.

If $B \neq 1$ then $0 \leq \zeta_{\nu_1} \leq 3 - \frac{\sigma}{2} - \frac{3}{2} < 0$; a contradiction.

If $B = 1$ then $0 \leq \zeta_{\nu_1} \leq 3 - \frac{\sigma}{2} - \frac{7}{2} < 0$; a contradiction.

Therefore, $p = 0$ and so by the formula, we obtain

$$0 \leq \zeta_{\nu_1} = \eta - \frac{\sigma}{2} + 3 \leq 3 - \nu_1.$$

Thus $\nu_1 = 3, \sigma = 6, \zeta_{\nu_1} = 0, g_0 = 25$; hence,

$$t_2 + t_3 = 8, \quad t_2 + 3t_3 = 25 - 3 = 22.$$

Then $t_2 = 1, t_3 = 7$ and the type is $[6 * 6; 3^7, 2]$ or its associates.

- Suppose that $g = 2$. Then $Z^2 = 2$ and the type is $[4 * 5; 2^{10}]$ or its associates , where $D^2 = 4g - 8 = 0$. This case has been already treated in Theorem 5.

- Suppose that $g = 1$. Then $Z^2 = 1$ and

1. if $D^2 = -4$ then the type is $[4 * 5; 2^{11}]$ or its associates , where $D^2 = 4g - 8 = -4$.
2. If $D^2 = -3$ then the type is $[6 * 6; 3^7, 2^3]$ or its associates , where $D^2 = 4g - 7 = -3$.
3. If $D^2 = -2$ then the type is $[8 * 8; 4^7, 3^2]$ or its associates , where $D^2 = 4g - 6 = -2$.

These case have been already treated in Theorem 5.

Theorem 8 *Suppose that $P_{2,1}[D] = 2$. Then $Z^2 = g$ and*

1. *if $D^2 = 4g - 8$, then the type is $[4 * 5; 2^r]$ or its associates, where $g = 12 - r > 0$.*
2. *If $D^2 = 4g - 7$, then the type is $[6 * 6; 3^7, 2^\varepsilon]$ or its associates, where $g = 4 - \varepsilon > 0$.*
3. *If $D^2 = 4g - 6$, then the type is $[8 * 8; 4^7, 3^2]$ or its associates.*

12 curves with $P_{2,1}[D] = 3$

Assume that $P_{2,1}[D] = 3$. Then $Z^2 = g + 1$ and hence, First , if the type is $[d; 1]$ then $d = 7, g = 15, Z^2 = 16$. Second, assume that (S, D) is derived from a $\#-$ minimal model.

Defining l to be $4g - D^2$, we obtain $D^2 = 4g - l$ and $K_S^2 = 5 + g - l$. From $K_S^2 = 8 - r$, it follows that $r = 3 + l - g$. Then by definition,

$$\xi_0 = 4 - \frac{l}{2}, \alpha = l - 4, \xi_2 = \sigma + f - 8 + \frac{B\sigma}{2} + 4 - \frac{l}{2}$$

We shall give an estimate of the magnitude of l.

Lemma 10 *If* $5+g \geq l$ *then* $l \geq 8$.

Proof: By $K_S^2 = 5+g-l \geq 0$, we have $|D-Z| = |-K_S| \neq \emptyset$. Hence, $(2Z-D)\cdot(Z-D) \leq 0$. Therefore,

$$2Z^2 - 3Z\cdot D + D^2 \leq 0.$$

Hence,

$$2(g+1) = 2Z^2 \leq 6\overline{g} - D^2,$$

and so $8 \leq l$. □

If $l \leq 6$ then applying the previous lemma, we get $5+g \geq 6 \geq l$ and thus, $l \geq 8$; a contradiction.

If $l = 7$ and $g \geq 2$ then $5+g \geq 7 = l$ and hence, $l \geq 8$; a contradiction.

Therefore, in the case when $l = 7$, we may assume that $g = 1$. Then $Z^2 = 2$ and $D^2 = -7$. By Theorem 8, the type is $[5*7, 1; 2^{13}]$.

When $l \geq 8$, we shall consider in the following two cases: A) case $\nu_1 \geq 3$ and B) case $\nu_1 \leq 2$.

12.0.1 A) case $\nu_1 \geq 3$

In order to study the case when $l \geq 8$, we begin with the case in which $\sigma \geq 6$. Then $|3Z-2D| \neq \emptyset$ by Theorem 1 and since $2Z-D$ is nef, it follows that

$$(3Z-2D)\cdot(2Z-D) \geq 0,$$

and hence,

$$6Z^2 - 7Z\cdot D + 2D^2 \geq 0.$$

By

$$6Z^2 - 7Z\cdot D + 2D^2 = 6(g+1) - 14\overline{g} + 2(4g-l) = 20 - 2l,$$

we obtain $l \leq 10$;hence, $l = 8, 9, 10$.

Moreover,

$$\begin{aligned} 0 \leq \zeta_{\nu_1} =& \eta + \xi_0\sigma + \alpha - \xi_2 p \\ =& \eta + (4 - \frac{l}{2})\sigma + l - 4 - \xi_2 p. \end{aligned}$$

To show that $p = 0$, we assume $p \geq 1$. Then $\sigma = p + 2\nu_1 \geq 7$ and since $l \geq 8$, it follows that

$$\begin{aligned}0 \leq \zeta_{\nu_1} <& \eta + 4\sigma + 4 - \frac{l\sigma - l}{2} + l - \sigma - f - \frac{B\sigma}{2} \\ =& \eta + 3\sigma - f + \frac{3l - \sigma l}{2} - \frac{B\sigma}{2}.\end{aligned}$$

First assume $B \neq 1$. Then by $\sigma \geq 7$,we get

$$\begin{aligned}0 \leq \zeta_{\nu_1} <& (3 - \frac{l}{2} - \frac{B}{2})\sigma + \frac{3l}{2} - f \\ \leq& 21 - 2l - \frac{7B}{2} - f.\end{aligned}$$

However, since $l \geq 8$, it follows that

$$21 - 2l - \frac{7B}{2} - f \leq 5 - \frac{7B}{2} - f \leq -2.$$

Second, assume that $B = 1$. Then recalling that $\sigma \geq 7, f \geq 3, B = 1$, we get

$$\begin{aligned}0 \leq \zeta_{\nu_1} \leq& (4 - \frac{l}{2})\sigma + (l - 4) + \eta - \xi_2 p \\ \leq& (4 - \frac{l}{2})\sigma + (l - 4) + (2 + \frac{l}{2} - f - \sigma)p \\ \leq& (4 - \frac{l}{2})\sigma + (l - 4) + (2 + \frac{l}{2} - f - \sigma) \\ =& \frac{3l}{2} - 2 - f + (3 - \frac{l}{2})\sigma \\ \leq& \frac{3l}{2} + 21 - 2 - f - \frac{7l}{2})\sigma \\ \leq& 19 - f - 2l \leq 0.\end{aligned}$$

Hence,

$$0 \leq \zeta_{\nu_1} \leq 19 - f - 2l \leq 0.$$

If $\zeta_{\nu_1} = 0$,then $l = 8, \sigma = 7, p = 1, \nu_1 = 3, \tilde{B} = 13, g_0 = 33$. By genus formula,

$$t_2 + t_3 = r = 11 - g, \quad t_2 + 3t_3 = r = g_0 - g = 33 - g.$$

Hence,$2t_2 = -2g$. This implies that $g = 0$.

This is a contradiction and thus $p = \sigma - 2\nu_1 = 0$ is checked. Therefore,

$$\zeta_{\nu_1} = \eta + (4 - \frac{l}{2})\sigma + l - 4$$

has been established.

12.1 case $D^2 = 4g - 8$

Then $l = 8, r = 3 + l - g = 11 - g$. If $g = 1$ then $Z^2 = 2, D^2 = 4g - 8 = -4$. This case has been already treated in Theorem 5. So we may assume $g \geq 2$. Since $\sigma = 2\nu_1$ we get

$$\zeta_{\nu_1} = \eta + 4.$$

12.1.1 case $\eta = 0$

If $\eta = 0$ then $\sigma - f - B\sigma/2 = 0$ and $\zeta_{\nu_1} = 4$; hence, we obtain the equation:

$$4 = F(\nu_1) = (\nu_1 - 3)x_1 + 2(\nu_1 - 4)x_2 + \cdots .$$

Then from $4 \geq \nu_1 - 3$, it follows that $\nu_1 \leq 7$.

• If $\nu_1 = 7$ then $x_1 = 1$ and hence, $\sigma = 14, g_0 = 13^2 = 169$ and $t_3 + t_6 = x_1 = 1, t_4 = t_5 = 0$, which yields

$$t_2 + t_3 + t_6 + t_7 = 11 - g, t_2 + 3t_3 + 15t_6 + 21t_7 = 169 - g.$$

Thus

$$t_2 + t_7 = 10 - g, 2t_3 + 14t_6 + 20t_7 = 158; 6t_6 + 10t_7 = 78, \quad 3t_6 + 5t_7 = 39, t_6 \leq 1.$$

This is impossible.

• If $\nu_1 = 6$, then $\sigma = 12, g_0 = 11^2 = 121$ and $4 = F(6) = 3x_1 + 4x_2$. Thus, $x_1 = 0, x_2 = 1$;hence, $t_3 = t_5 = 0, t_4 = 1$ and so

$$t_2 + t_4 + t_6 = 11 - g, \quad t_2 + 6t_4 + 15t_6 = 121 - g.$$

Accordingly,

$$5t_4 + 14t_6 = 110.$$

Thus $14t_6 = 110 - 5t_4 = 105$; a contradiction.

• If $\nu_1 = 5$, then $\sigma = 10, g_0 = 9^2 = 81, 4 = F(5) = 2x_1$. Thus, $x_1 = 2$;hence, $t_3 + t_4 = 2$ and therefore,

$$t_2 + t_3 + t_4 + t_5 = 11 - g, \quad t_2 + 3t_3 + 6t_4 + 10t_5 = 81 - g.$$

Hence,

$$2t_3 + 5t_4 + 9t_5 = 70, \quad t_4 + 3t_5 = 22.$$

Then $t_4 = 1, t_3 = 1, t_5 = 7, t_2 = \varepsilon$; thus $g = 2 - \varepsilon$ and the type is $[10 * 10; 5^7, 4, 3, 2^\varepsilon]$ or its associates.

• If $\nu_1 = 4$, then $\sigma = 8, g_0 = 7^2 = 49, \zeta_{\nu_1} = 4$ and $\zeta_{\nu_1} = t_3$, i.e. $t_3 = 4$. Hence,

$$t_2 + t_3 + t_4 = 11 - g, \quad t_2 + 3t_3 + 6t_4 = 49 - g.$$

Hence, $2t_3 + 5t_4 = 38; t_4 = 6, t_2 = 1 - g$. Thus $g = 1$ and the type is $[8 * 8; 4^6, 3^4]$ or its associates.

12.1.2 case $\eta \neq 0$

If $\eta \neq 0$ then $\eta = -4, \nu_1 = 3, 4; \zeta_{\nu_1} = 0$ and therefore, we have two cases: $(1)\nu_1 = 3$ and (2) $\nu_1 = 4$.

• If $\nu_1 = 3$, then from $\eta = -4, \eta = 2(\nu_1 - 2)(4\nu_1 - 2\nu_1 B - f)$, it follows that $3B + f = 8$ and so $g_0 = 35$. By genus formula

$$t_2 + t_3 = 11 - g, \quad t_2 + 3t_3 = 35 - g.$$

Hence, $2t_3 = 24, t_3 = 12, t_2 < 0$; a contradiction.

• If $\nu_1 = 4$, then by the same argument as before, $\zeta_4 = 0, t_3 = 0, (2 - B)4 - f = -1$ and $g_0 = 56$. By genus formula

$$t_2 + t_4 = 11 - g, \quad t_2 + 6t_4 = 56 - g.$$

Hence, $5t_4 = 45; t_4 = 9$. And hence $t_2 = 2 - g$ and the type is $[8*9; 4^9, 2^\varepsilon], g = 2 - \varepsilon$ or its associates, where $D^2 = 4g - 8$.

12.2 case $D^2 = 4g - 9$

Suppose that $D^2 = 4g - 9$. Then $l = 9$ and $r = 12 - g$.

Therefore,

$$0 \leq \zeta_{\nu_1} = \eta - \nu_1 + 5 \leq -\nu_1 + 5.$$

• case $\nu_1 = 5$ Then $\zeta_{\nu_1} = 0, \sigma = 10$ and $\eta = 0$. Hence, $t_3 = t_4 = 0$ and $g_0 = 81$. By genus formula

$$t_2 + t_5 = 12 - g, \quad t_2 + 10t_5 = 81 - g.$$

Hence, $9t_5 = 81 - 12 = 69$; a contradiction.

• case $\nu_1 = 4$ Then $\sigma = 8, 0 \leq \zeta_{\nu_1} = \eta + 1; -1 \leq \eta$. Moreover,

$$\eta = (\sigma - 4)(\sigma - f - B\sigma/2) = 4(8 - f - 4B) = -4, 0.$$

Hence, $\eta = 0$ and thus $\zeta_{\nu_1} = t_3 = 1, g_0 = 49$. By genus formula,

$$t_2 + t_3 + t_4 = 12 - g, \quad t_2 + 3t_3 + 6t_4 = 49 - g.$$

Hence, $5t_4 = 35, t_4 = 7$ and $t_2 = 4 - g$. The type is $[8 * 8; 4^7, 3, 2^{4-g}]$ or its associates, where $g = 1, 2, 3, 4$.

• case $\nu_1 = 3$. Then $\zeta_{\nu_1} = 0$ and $\sigma = 6, \zeta_{\nu_1} = \eta + 2$; thus $\eta = -2$. Hence, $f + 3B = 7$ and $g_0 = 30$. By genus formula,

$$t_2 + t_3 = 12 - g, \quad t_2 + 3t_3 = 30 - g.$$

Hence, $t_3 = 9, t_2 = 3 - g$. The type is $[6 * 7; 3^9, 2^{3-g}]$ or its associates where $g = 1, 2, 3$.

12.3 case $D^2 = 4g - 10$

Suppose that $D^2 = 4g - 10$. Then $l = 10, r = 13 - g$,

$$0 \leq \zeta_{\nu_1} = \eta - 2\nu_1 + 6 \leq -2(\nu_1 - 3).$$

Hence, $\nu_1 = 3, \sigma = 6$ and $\eta = 0, 3B + f = 6$. Clearly, $g_0 = 25$.

By genus formula,

$$t_2 + t_3 = 13 - g, \quad t_2 + 3t_3 = 25 - g.$$

Hence, $t_3 = 6$ and $t_2 = 7 - g$. The type is $[6 * 6; 3^6, 2^{7-g}]$ or its associates, where $g = 1, 2, \cdots, 7$.

12.3.1 B) case $\nu_1 \leq 2$

Since $\nu_1 \leq 2$, it follows that

$$4 = 2(g+1) - 2\overline{g} = 2Z^2 - D \cdot Z = (2Z - D) \cdot Z = \tau_3 - 2.$$

Hence, $\tau_3 = 6$. From

$$(\sigma - 3)(2f + B\sigma - 6) = 6,$$

we obtain either (1) $\sigma - 3 = 1, 2f + B\sigma - 6 = 6$ or (2) $\sigma - 3 = 2, 2f + B\sigma - 6 = 3$.

case (1) $\sigma = 4, 2f + B\sigma = 12, g_0 = 15$. The type is $[4 * 6; 2^r]$ and its associates , where $g = 15 - r = 1, 2, \cdots, 14$ and $D^2 = 4g - 12$.

case (2) $\sigma = 5, 2f + B\sigma = 9, g_0 = 14$ and the type is $[5 * 7, 1; 2^r]$, where $g = 14 - r$ and $D^2 = 4g - 11$.

Accordingly, we establish the following result:

Theorem 9 *Suppose that $P_2[D] = 3g > 1$. Then $Z^2 = g + 1$ and*

- *case $S = \mathbf{P}^2$. Then the type is $[7; 1]$ and $g = 15, D^2 = 49$.*
- *case $\nu_1 \leq 2$. Then the type is (1) $[4 * 6; 2^r]$ or its associates , where $g = 15 - r$ and $D^2 = 4g - 12$, or (2) $[5 * 7, 1; 2^r]$, where $g = 14 - r$ and $D^2 = 4g - 11$.*
- *case $\nu_1 \geq 3$. Then*

1. *if $5 \geq g \geq 7$ then the type is $[6 * 6; 3^6, 2^{7-g}]$ or its associates , where $D^2 = 4g - 10$.*
2. *If $g = 4$ then*
   - *(a) if $D^2 = 7$ then the type is $[8 * 8; 4^7, 3]$ or its associates.*
   - *(b) If $D^2 = 6$ then the type is $[6 * 6; 3^6, 2^3]$ or its associates.*
3. *If $g = 3$ then*
   - *(a) if $D^2 = 2$ then the type is $[6 * 6; 3^6, 2^4]$ or its associates.*
   - *(b) If $D^2 = 3$ then the type is $[8 * 8; 4^7, 3, 2]$ or $[6 * 7; 3^9]$ or their associates.*
4. *If $g = 2$ then*
   - *(a) if $D^2 = 0$ then the type is either $[10 * 10; 5^7, 4, 3]$ or $[8 * 9; 4^9]$ or their associates.*

(b) *If* $D^2 = -1$ *then the type is either* $[6 * 7; 3^9, 2]$ *or* $[8 * 8; 4^7, 3, 2^2]$ *or their associates.*

(c) *If* $D^2 = -2$ *then the type is* $[6 * 6; 3^6, 2^5]$ *or its associates.*

5. *If* $g = 1$ *then*

   (a) *if* $D^2 = -4$ *then the type is* $[8 * 8; 4^6, 3^4]$ *or* $[8 * 9; 4^9, 2]$ *or* $[10 * 10; 5^7, 4, 3, 2]$ *or their associates.*

   (b) *If* $D^2 = -5$ *then the type is* $[6*7; 3^9, 2^2]$ *or* $[8*8; 4^7, 3, 2^3]$ *or their associates.*

   (c) *If* $D^2 = -6$ *then the type is* $[6 * 6; 3^6, 2^6]$ *or its associates.*

When $\sigma = 3$, the invariants are easily computed:

$$A = Z^2 - \overline{g} = -1, \alpha = \overline{g} - 9, \omega = -9, \Omega = -3 - \overline{g}.$$

Moreover if the type is $[d; 1], d \geq 4$, then

$$A = \frac{(d-3)(d-6)}{2}, \alpha = d(d-6), \omega = \frac{d(d-9)}{2}, \Omega = (d-3)(d-9).$$

13 curves with $Q = 1, 2$

Here, Q denotes $(2Z - D)^2$.

Proposition 10 *Assume that* $Q = 1$. *Then*

1. (S, D) *is obtained from a plane curve of degree 7 with at most double points and* $g = 15 - r \leq 15$ *or*

2. *the type is* $[6 * 6; 3^7, 2^\varepsilon]$ *or its associates, where* $g = 4 - \varepsilon$ *or*

3. *the type is* $[5; 1]$ *or*

4. *the type is* $[3 * 5, 1; 1]$.

Assume that $Q = 2$. *Then*

1. *the type is* $[8 * 8; 4^7, 3^2]$ *or its associates or*

2. *the type is* $[6 * 6; 3^6, 2^\varepsilon]$ *or its associates , where* $g = 7 - \varepsilon$ *and* $D^2 = 4g - 10$ *or*

3. *the type is* $[5{*}5;2^r]$ *or* $[5{*}10,2;2^r]$ *or their associates , where* $g = 16-r$ *and* $D^2 = 50 - 4r = 4g - 14$ *or*

4. *the type is* $[3 * 3; 1]$.

Proof: First suppose that $\nu_1 \geq 3$. By $Q = (2Z - D)^2$, we get

$$\begin{aligned} 0 &\leq (3Z - 2D) \cdot (2Z - D) \\ &= (4Z - 2D) \cdot (2Z - D) - Z \cdot (2Z - D) \\ &= 2w - Z \cdot (2Z - D). \end{aligned}$$

Since $1 \leq Z \cdot (2Z - D)$, it follows that

$$0 \leq (3Z - 2D) \cdot (2Z - D) = 2w - Z \cdot (2Z - D) < 2w.$$

Suppose that $Q = 1$. Then $Z\cdot(2Z-D) = 2$;hence, $Z^2 = g, D^2 = 4g-7$. By Theorem 8, the type is $[6 * 6; 3^7, 2^\varepsilon]$ and $g = 4 - \varepsilon$.

Suppose that $Q = 2$. Then we have two cases (1) $Z \cdot (2Z - D) = 2$ and (2) $Z \cdot (2Z - D) = 4$.

case (1) $Z \cdot (2Z - D) = 2$. Then, $Z^2 = g$ and $D^2 = 4g - 6$. By Theorem 8 the type is $[8 * 8; 4^7, 3^2]$ or its associates, where $g = 1$.

case (2) $Z\cdot(2Z-D) = 4$. Then, $Z^2 = g+1, D^2 = 4g-10$. By Theorem 9, the type is $[6 * 6; 3^6, 2^\varepsilon]$ or its associates , where $g = 7 - \varepsilon > 0$.

Second,suppose that $\nu_1 \leq 2$ and (S, D) is obtained from (Σ_B, C) which is # minimal. Then $Q = (2Z - D)^2 = \tau_4$. Note that

$$\tau_4 = (\sigma - 4)(B\sigma + 2f - 8).$$

If $Q = 1$, then either 1) $\sigma - 4 = 1, B = 1, f = 2$ and the type is $[5 * 7, 1; 2^r]$ where $g_0 = 24 - 10 = 14, g = 14 - r$ or 2) $\sigma - 4 = -1, B = 1, f = 2$ and the type is $[3 * 5, 1; 1]$ where $g_0 = g = 9$.

If $Q = 2$, then $\sigma - 4 = i$ and $B(i + 4) + 2f - 8 = \dfrac{2}{i}$.

When $B = 0$, we obtain either 1) $i = 1, \sigma = f = 5$ and the type is $[5 * 5; 2^r]$ and $g_0 = 16$, or 2) $i = -1, \sigma = f = 3$ and the type is $[3 * 3; 1]$ and $g_0 = g = 4$,

When $B = 1$, we obtain $i + 2f = \dfrac{2}{i}$. This case cannot occur.

When $B \geq 2$, we obtain $i = 1, f = 0, B = 2, \sigma = f = 5$. Thus the type is $[5 * 10, 2; 2^r]$ and $g = 16 - r$.

Finally, suppose that the type of (S, D) is $[d; 1]$. From $2Z-D = (d-6)H$, it follows that $(d-6)^2H^2 = Q = 1, 2$. Then $Q = 1$ and $d = 5$ or $d = 7$. □

13.1 Formula II'

Since

$$D + \nu_1 K_S \sim C + \nu_1 K_0 + \sum_{j=1}^{r} (\nu_1 - \nu_j) E_j,$$

it follows that

$$(D+\nu_1 K_S)\cdot(D+2K_S) = (\nu_1 Z_0 - (\nu_1 - 1)C)\cdot(2Z_0 - C) + \sum_{j=1}^{r} (\nu_1 - \nu_j)(\nu_j - 2).$$

Put

$$\rho_{\nu_1} = (D + \nu_1 K_S) \cdot (D + 2K_S), \quad \theta_{\nu_1} = (\nu_1 Z_0 - (\nu_1 - 1)C) \cdot (2Z_0 - C),$$

$$\zeta_{\nu_1} = \sum_{j=1}^{r} (\nu_1 - \nu_j)(\nu_j - 2).$$

Making use of the symbol t_j which denotes the number of $j-$ ple singular points of the curve C, ζ_{ν_1} can be rewritten as follows:

$$\zeta_{\nu_1} = \sum_{j=3}^{\nu_1 - 1} (\nu_1 - j)(j - 2) t_j.$$

By Lemma 3, we obtain the next result:

Lemma 11 (Formula II') *Let* $\rho_{\nu_1} = (D + \nu_1 K_S) \cdot (D + 2K_S)$. *Then*

$$\rho_{\nu_1} = 2\nu_1 {K_S}^2 - (\nu_1 + 1)D^2 + 2(2 + \nu_1)\overline{g},$$

and

$$\rho_{\nu_1} = \zeta_{\nu_1} + \theta_{\nu_1}, \quad \theta_{\nu_1} = \tilde{A}(\sigma - 2\nu_1) + \gamma$$

where $\tilde{A} = (\sigma + \nu_1 - 2)B + 2f - 2\nu_1 - 4$ *and* $\gamma = 2(\nu_1 - 2)(f + \nu_1 B - 2\nu_1)$.

Corollary 5 *If* $p = \sigma - 2\nu_1 > 0$, *then* $\tilde{A} + \gamma \geq 3\nu_1 - 5$. *Moreover, if* $\tilde{A} + \gamma = 3\nu_1 - 5$, *then* $B = 1$ *and* $\sigma - 2\nu_1 = 1$.

Proof: If $B = 0$ then $\tilde{A} = 2f - 2\nu_1 - 4 \geq 2(p + 2\nu_1) - 2\nu_1 - 4 \geq 2\nu_1 - 2$ and $\dfrac{\gamma}{2\nu_1 - 4} = f - 2\nu_1 \geq p \geq 1$. Hence, $\tilde{A} + \gamma \geq 2\nu_1 - 2 + 2\nu_1 - 4 = 4\nu_1 - 6$.

If $B = 1$ then

$$\tilde{A} = \sigma + \nu_1 - 2 + 2f - 2\nu_1 - 4 = p + 2\nu_1 + \nu_1 - 2 + 2f - 2\nu_1 - 4 \geq 3\nu_1 - 5$$

and $\frac{\gamma}{2\nu_1 - 4} = f + \nu_1 - 2\nu_1 \geq 0$. Hence, in particular, $\tilde{A} + \gamma \geq 3\nu_1 - 5$.

If $B \geq 2$ then

$$\tilde{A} \geq 2(\sigma + \nu_1 - 2) + 2f - 2\nu_1 - 4 \geq 4\nu_1 - 6 \text{and} \frac{\gamma}{2\nu_1 - 4} = f + (B-2)\nu_1 \geq 0.$$

□

Lemma 12

$$(\nu_1 Z - (\nu_1 - 1)D) \cdot (2Z - D) = \tau_{\nu_1+2} - 2(\nu_1 - 2)^2 + \zeta_{\nu_1}.$$

Proof: From

$$(\nu_1 Z - (\nu_1 - 1)D) \cdot Z = \tau_{\nu_1+1} - 2(\nu_1 - 1)^2 + \widetilde{\delta}(\nu_1),$$

$$(\nu_1 Z - (\nu_1 - 1)D) \cdot D = \tau_{\nu_1} - 2\nu_1^2 + \widetilde{\delta}_0(\nu_1),$$

and

$$\zeta_{\nu_1} = 2\widetilde{\delta}(\nu_1) - \widetilde{\delta}_0(\nu_1),$$

it follows that

$$\begin{aligned}\rho_{\nu_1} =& (\nu_1 Z - (\nu_1 - 1)D) \cdot (2Z - D) \\ =& 2\tau_{\nu_1+1} - 4(\nu_1 - 1)^2 + 2\widetilde{\delta}(\nu_1) - (\tau_{\nu_1} - 2\nu_1^2 + \widetilde{\delta}_0(\nu_1)) \\ =& 2\tau_{\nu_1+1} - \tau_{\nu_1} - 2(\nu_1 - 2)^2 + 4 + \zeta_{\nu_1} \\ =& \tau_{\nu_1+2} - 2(\nu_1 - 2)^2 + \zeta_{\nu_1}.\end{aligned}$$

□

In particular, $\theta_{\nu_1} = \tau_{\nu_1+2} - 2(\nu_1 - 2)^2$.

14 rational curves

In what follows, we shall study minimal pairs (S, D) with $\kappa[D] = 2$ and $g(D) = 0$. In this case, $\sigma \geq 4, \beta = -D^2 \geq 5$ and $K_\beta = K_S + (1 - \frac{2}{\beta})D$ is nef and big. Moreover, $P_2[D] \geq 2$ and $K_\beta^2 = K_S^2 - \beta + 4 - \frac{4}{\beta} > 0$.

Since $\sigma \geq 4$, the next result has been proved in Proposition 3 for non-rational curves.

Lemma 13 *If $g(D) = 0$ then $2Z - D = D + 2K_S$ is nef.*

Proof: First note that $(D + 2K_S) \cdot D = -\beta + 2(\beta - 2) = \beta - 4 \geq 1$. If there exists an irreducible curve $A \neq D$ such that $(D + 2K_S) \cdot A < 0$, then $A^2 < 0, A \cdot K_S < -A \cdot D/2 \leq 0$. Hence, A turns out to be an exceptional curve and $A \cdot D < -2A \cdot K_S = 2$; thus $A \cdot D < 2$. This contradicts the minimality of (S, D). ▭

Lemma 14 *${K_S}^2 \leq -1$ and $Z^2 \leq \beta - 5$.*

Proof: Suppose that $8 - r = {K_S}^2 \geq 0$. Then by Riemann-Roch, $|-K_S| \neq \emptyset$. Since $D + 2K_S$ is nef, it follows that $(D + 2K_S) \cdot K_S \leq 0$ and so

$$(D + 2K_S) \cdot K_S = \beta - 2 + 2(8 - r) \leq 0.$$

Hence,

$$\beta - 2 + 2(8 - r) = 14 - 2r + \beta \leq 0,$$

thus $7 + \frac{5}{2} \leq 7 + \frac{\beta}{2} \leq r$. Hence, $10 \leq r$. This contradicts the inequality $8 - r = {K_S}^2 \geq 0$.

Moreover, from $(Z - D)^2 = {K_S}^2 \leq -1$, the result follows immediately.

▭

Proposition 11 *If $g(D) = 0$ then $Q = 4Z^2 - 8 - \beta = 4{K_S}^2 + 3\beta - 8 \geq 0$. Moreover, $4Z^2 - 8 - \beta = 0$ if and only if $\sigma = 4$.*

Proof: Since $2Z - D$ is nef and $|2Z - D| \neq \emptyset$, it follows that $Q = (2Z - D)^2 \geq 0$ and $Q = 4Z^2 - 8 - \beta \geq 0$.

Suppose that $\sigma = 4$. Then $\nu_1 \leq 2$ and $2Z - D = D + 2K_S = C + 2K_0 \sim (f - 4 + 2B)F_c$ and hence, $Q = 0$.

Next, under the hypothesis $Q = (2Z - D)^2 = 4{K_S}^2 + 3\beta - 8 = 0$, we shall derive $\sigma = 4$, examining the following cases, separately.

• case $\nu_1 \geq 3$. Then

$$(3Z - 2D) \cdot (2Z - D) = (D + 3K_S) \cdot (D + 2K_S) \geq 0.$$

On the other hand, $2(2Z - D) = Z + (3Z - 2D)$ and so

$$0 = 2Q = 2(2Z - D)^2 = 2(2Z - D) \cdot (2Z - D) = Z \cdot (2Z - D) + (3Z - 2D) \cdot (2Z - D) \geq 0.$$

Hence, $Z \cdot (2Z - D) = (3Z - 2D) \cdot (2Z - D) = 0$. Thus $D \cdot (2Z - D) = 0$, which implies that $\beta = -D^2 = -2D \cdot Z = 4$; a contradiction.

• case $\nu_1 \leq 2$. Then

$$0 = Q = \tau_4, \quad \tau_4 = (\sigma - 4)(\sigma B + 2f - 8),$$

$$0 = (\sigma - 4)(\sigma B + 2f - 8).$$

This implies that $\sigma - 4 = 0$. $\square$

Later, pairs (S, D) with $Q = 1, 2$ will be enumerated.

Proposition 12 *If D is a rational curve with $\kappa[D] = 2$, then $P_2[D] = Z^2 + 2$.*

Proof: Since K_β is nef and big and $\lceil K_\beta \rceil = D + K_S$, it follows that $H^1(S, \mathcal{O}_S(D + 2K_S)) = 0$ by a theorem of Kawamata. Hence, by Riemann-Roch,

$$\begin{aligned}\dim H^0(S, \mathcal{O}_S(D + 2K_S)) &= \frac{(D + K_S) \cdot (D + 2K_S)}{2} + 1 \\ &= \frac{Z \cdot (2Z - D)}{2} + 1 = Z^2 + 2.\end{aligned}$$

By $2(D + K_S) \cdot D < 0$, we get $|2Z| = |2Z - D| + D$; hence,

$$P_2[D] = \dim H^0(S, \mathcal{O}_S(D + 2K_S)) = Z^2 + 2.$$

This implies that $Z^2 \geq 0$, for $P_2[D] \geq 2$. $\square$

In later sections, pairs with $P_2[D] = 2, 3$ will be enumerated.

15 logarithmic plurigenera

However, logarithmic m genera are a little hard to compute.

Lemma 15 *If F is an effective divisor such that $F \cdot D < 0$ where D is an irreducible curve with $\beta = -D^2 > 0$, then letting $a_1 = \lceil \frac{F \cdot D}{\beta} \rceil$, $a_1 D$ becomes a fixed component of $|F|$.*

Further, $\dim |F| = \dim |F - a_1 D|$.

Proof: There exist an effective divisor F_1 which does not contain D and a positive integer a such that $F = F_1 + aD$. Since $F_1 \cdot D \geq 0$, $F \cdot D = F_1 \cdot D + aD^2 = F_1 \cdot D - a\beta \geq -a\beta$. Hence, $a \geq \frac{-F \cdot D}{\beta}$. Therefore, we obtain $a \geq a_1 = \lceil \frac{-F \cdot D}{\beta} \rceil$. $\square$

For $m \geq 2$, let $Y = (m-1)K_\beta = (m-1)K_S + (m-1)(1 - \frac{2}{\beta})D$, which is nef and big. Then $\lceil Y \rceil = (m-1)K_S + \lceil (m-1)(1 - \frac{2}{\beta}) \rceil D$ and by a theorem of Kawamata, $H^1(S, \mathcal{O}_S(K_S + \lceil Y \rceil)) = 0$.

Applying Lemma 15 to $F = mZ$, we obtain $F \cdot D = -2m$, $a_1 = \lceil \frac{2m}{\beta} \rceil$ and $K_S + \lceil Y \rceil = mK_S + \lceil (m-1)(1-\frac{2}{\beta}) \rceil D$.

Claim 5

$$mZ - \lceil \frac{2m}{\beta} \rceil D \leq mK_S + \lceil (m-1)(1-\frac{2}{\beta}) \rceil D \leq mZ.$$

Proof: It suffices to verify the inequalities:

$$m - \lceil \frac{2m}{\beta} \rceil \leq \lceil (m-1)(1-\frac{2}{\beta}) \rceil \leq m.$$

Let $q = [\frac{2m-2}{\beta}]$ and $2m-2 = q\beta + r_0$. Then $\lceil (m-1)(1-\frac{2}{\beta}) \rceil = m-q-1$ and $\lceil \frac{2m}{\beta} \rceil = \lceil \frac{2+q\beta+r_0}{\beta} \rceil = q+1$ or $q+2$. Hence, $m - \lceil \frac{2m}{\beta} \rceil = m-q-2$ or $m-q-1$. $\square$

Therefore,

$$\dim |mZ| = \dim |mZ - aD| = \dim |K_S + \lceil Y \rceil|.$$

Letting $V = \lceil Y \rceil$, we get $V = (m-1)Z - qD$ and $K_S + V = mZ - (q+1)D$. By a vanishing theorem of Kawamata, $H^1(S, \mathcal{O}_S(K_S+V)) = 0$ and so by Riemann-Roch,

$$\begin{aligned}
\dim |mZ| &= \dim |K_S + V| = \frac{V \cdot (K_S+V)}{2} \\
&= \frac{((m-1)Z - qD) \cdot (mZ - (q+1)D)}{2} \\
&= \frac{m(m-1)Z^2 + q(q+1)D^2 - (qm + (m-1)(q+1))Z \cdot D}{2} \\
&= \frac{m(m-1)Z^2 + (q+1)(-q\beta) + 2(qm + (m-1)(q+1))}{2} \\
&= \frac{m(m-1)Z^2}{2} + mq + \frac{r_0(q+1)}{2}.
\end{aligned}$$

Thus we establish the following result.

Proposition 13 *If D is a rational curve with $\beta = -D^2$ and $\kappa[D] = 2$ then, letting $q = [\frac{2m-2}{\beta}]$ and $2m-2 = q\beta + r_0$, we obtain*

$$P_m[D] = \frac{m(m-1)Z^2}{2} + mq + \frac{r_0(q+1)}{2} + 1.$$

In particular,

$$P_3[D] = 3Z^2 + 3.$$

When $m = 4$, we get $2m - 2 = 6 = q\beta + r_0$. If $\beta > 6$ then $q = 0, r_0 = 6$. Hence,

$$P_4[D] = 6Z^2 + 4.$$

If $\beta = 6$ then $q = 1, r_0 = 0$. Hence,

$$P_4[D] = 6Z^2 + 5.$$

If $\beta = 5$ then $q = 1, r_0 = 1$ and in this case $Z^2 = 0$. Hence,

$$P_4[D] = 6Z^2 + 6 = 6.$$

15.1 invariant $P_{3,1}[D]$

By Lemma 13, if $\sigma > 4$ then $2Z - D$ is nef and big. Hence, $H^1(S, \mathcal{O}_S(K_S + 2Z - D)) = 0$. Noting that $K_S + 2Z - D = 3Z - 2D \sim D + 3K_S$, by Riemann-Roch, we get

$$\dim H^0(S, \mathcal{O}_S(3K_S + D)) = \frac{(3Z - 2D) \cdot (2Z - D)}{2} + 1 = 3Z^2 + 8 + D^2.$$

If $\sigma < 6$ then $(3Z - 2D) \cdot F_c = (\sigma - 6)\Delta_0 \cdot F_c = (\sigma - 6) < 0$. Hence, $|3Z - 2D| = \emptyset$, i.e, $P_{3,1}[D] = 0$. Thus, we obtain the next result.

Proposition 14 *If D is rational, $\kappa[D] = 2$ and $\sigma > 4$, then*

$$P_{3,1}[D] = 3Z^2 + 8 + D^2.$$

Moreover, if $\sigma = 5$ then $P_{3,1}[D] = 0$.

Note that $P_{3,2}[D] = P_3[D] = 3Z^2 + 3$ and that if $\sigma \geq 6$ then $3Z^2 + 8 \geq \beta$.

Next, let Y be $\frac{3}{2}(2Z - D)$, that is nef and big. Hence, $H^1(S, \mathcal{O}_S(K_S + \lceil Y \rceil)) = 0$. However, $\lceil Y \rceil = 3Z - D$ and $K_S + \lceil Y \rceil = 4Z - 2D$. Hence,

$$\dim H^0(S, \mathcal{O}_S(4Z - 2D)) = \frac{(4Z - 2D) \cdot (3Z - D)}{2} + 1 = 6Z^2 + 11 + D^2.$$

Thus, we obtain the next result.

Proposition 15 *If D is rational, $\kappa[D] = 2$ and $\sigma > 4$, then*

$$P_{4,2}[D] = 6Z^2 + 11 + D^2.$$

16 curves with $P_2[D] = 2$

We shall give a complete list of types of pairs (S, D) such that $P_2[D] = 2, \kappa[D] = 2, g(D) = 0$. Hence, suppose that $\kappa[D] = 2, g(D) = 0, P_2[D] = 2$. Then by Lemma 12, $Z^2 = 0$; i.e. ${K_S}^2 - D^2 = 4$ and so $\beta = r - 4 \geq 5$. Note that

$$\rho_{\nu_1} = (\nu_1 Z - (\nu_1 - 1)D) \cdot (2Z - D) = (6 - \beta)\nu_1 + \beta - 4.$$

Moreover, from $Q \geq 0$, it follows that

$$Q = 4Z^2 - 4Z \cdot D + D^2 = 8 - \beta.$$

Hence, we have four cases according to the value of β, i.e. $\beta = 5, 6, 7, 8$.

However, first we shall consider the case when $\nu_1 = 2$.

Proposition 16 *If $Z^2 = 0$ and $\nu_1 = 2$, then $\beta = 12$.*

Proof: From $2Z^2 - 2(g - 1) = 2Z^2 - Z \cdot D = 2{Z_0}^2 - Z_0 \cdot C = \tau_3 - 2$, it follows that

$$(\sigma - 3)(\sigma B + 2f - 6) = \tau_3 = 2 + 2Z^2 - 2(g - 1) = 4.$$

Hence, $\sigma - 3 = 1, 2$.

If $\sigma = 4$, then $\sigma B + 2f - 4 = 2 + 4 = 6$, thus $(B, f) = (0, 5), (1, 3), (2, 1)$. In each case, $g_0 = 12, r = 12, {K_S}^2 = -4, \beta = 12$. The type is $[4 * 5; 2^{12}]$ or its associates.

If $\sigma = 5$, then $5B + 2f - 4 = 2 + 2 = 4$, thus $(B, f) = (0, 4)$. But this is impossible, for $\sigma \leq f$. □

Second, under the hypothesis $\nu_1 \geq 3$ we shall examine the following four cases, $\beta = 5, 6, 7, 8$, separately.

16.1 case $\beta = 5$

Then $r = 9$ and $K_S{}^2 = -1$ and moreover $\rho_{\nu_1} = \nu_1 + 1$; hence, by Formula II' (Lemma 11), $\nu_1 + 1 = \zeta_{\nu_1} + \theta_{\nu_1}$.

Claim 6 $p = \sigma - 2\nu_1 = 0$.

Proof: Otherwise, from $\theta_{\nu_1} \geq \tilde{A}+\gamma \geq 3\nu_1 - 5$, it follows that $\nu_1+1 \geq 3\nu_1-5$. Hence, $\nu_1 \leq 3$; i.e. $\nu_1 = 3$. Thus $B = 1, \sigma = 2\nu_1 + 1 = 7, f = \nu_1 = 3$; hence, $g_0 = 6 \cdot 9 - 21 = 33$. Therefore, by genus formula,

$$t_2 + t_3 = 9, \quad t_2 + 3t_3 = 33.$$

Thus $2t_3 = 33 - 9 = 24 > 18$; a contradiction. $\square$

Therefore $p = 0$ has been established and

$$\nu_1 + 1 = \zeta_{\nu_1} + 2(\nu_1 - 2)(f + \nu_1 B - 2\nu_1).$$

Letting $q = f + \nu_1 B - 2\nu_1$, we get $\nu_1 + 1 \geq 2q(\nu_1 - 2)$ and thus

$$3 \leq \nu_1 \leq \frac{4q+1}{2q-1} = 2 + \frac{3}{2q-1}.$$

Hence, if $q > 0$ then $q = 1, 2$. Note that $\gamma = 2q(\nu_1 - 2)$

16.1.1 case $\gamma > 0$

If $q = 1$, then $\nu_1 \leq 5$. If $q = 2$, then $\nu_1 \leq 3$.

• $\nu_1 = 5$. Then $\sigma = 10, q = 1$.

From $1 = f + \nu_1 B - 2\nu_1$, it follows that $g_0 = 90$ and $\zeta_{\nu_1} = \zeta_5 = 0$. Hence, $t_3 = t_4 = 0$. From $t_2 + t_5 = r = 9$ and $t_2 + 10t_5 = g_0 = 90$, it follows that $t_2 = 0, t_5 = 9$. The type is $[10 * 11; 5^9]$ or its associates.

• case $\nu_1 = 4$. Then $\sigma = 8$ and $q = 1$.

From $1 = f + \nu_1 B - 2\nu_1$, it follows that $g_0 = 56$ and $\zeta_{\nu_1} = \zeta_4 = 1$. Thus $t_3 = 1$. From $t_2 + t_3 + t_4 = r = 9, \quad t_2 + 3t_3 + 6t_4 = g_0 = 56$, it follows that $t_4 = 9$ and $t_2 = -1$; a contradiction.

• case $\nu_1 = 3$. Then $\sigma = 6, \zeta_{\nu_1} = \zeta_3 = 0, 4 = \zeta_{\nu_1} + 2q$. Hence, $q = 2$. From $2 = f + \nu_1 B - 2\nu_1$, it follows that $\sigma = 6$ and $g_0 = 35$. From $t_2 + t_3 = r = 9, \quad t_2 + 3t_3 = g_0 = 35$, it follows that $2t_3 = 26, t_2 < 0$; a contradiction.

16.1.2 case $\gamma = 0$

Finally, we consider the case in which $q = 0$, i.e. $\gamma = 0$. Then $\theta_{\nu_1} = 0$ and from $\theta_{\nu_1} = (\nu_1 - 2)(\tilde{B} - 2\sigma)$ it follows that $\tilde{B} - 2\sigma = 0$; hence, $2g_0 = \tau_1 = (\sigma - 1)(\tilde{B} - 2) = 2(\sigma - 1)^2$.

Moreover, $\nu_1 + 1 = \zeta_{\nu_1}$; hence,

$$\nu_1 + 1 = F(\nu_1) = (\nu_1 - 3)x_1 + 2(\nu_1 - 4)x_2 + 3(\nu_1 - 5)x_3 + \cdots .$$

Since $\nu_1 + 1 = F(\nu_1) \geq 2(\nu_1 - 4)$ it follows that $\nu_1 \leq 9$.

• case $\nu_1 = 9$ In this case, $\sigma = 18, g_0 = 17^2 = 289$ and

$$\nu_1 + 1 = 10 = F(9) = 6x_1 + 10x_2 + \cdots$$

Then $x_2 = 1, x_1 = x_3 = \cdots = 0$ and since $t_4 + t_7 = x_2 = 1$, it follows that

$$t_2 + t_4 + t_7 + t_9 = 9, \quad t_2 + 6t_4 + 21t_7 + 36t_9 = 289.$$

Therefore,

$$5t_4 + 20t_7 + 35t_9 = 280, \quad t_4 + 4t_7 + 7t_9 = 56, \quad 7t_9 = 55 - 3t_7 = 55 \text{ or } 52.$$

This is a contradiction.

• case $\nu_1 = 8$

In this case $\sigma = 16, g_0 = 15^2 = 225$ and

$$\nu_1 + 1 = 9 = F(8) = 5x_1 + 8x_2 + 9x_3.$$

Hence, $t_5 = x_3 = 1, t_3 = t_4 = t_7 = t_6 = 0$. By genus formula

$$t_2 + t_5 + t_8 = 9, t_2 + 10t_5 + 28t_8 = 225.$$

From these, we get $t_5 + 3t_8 = 23$; a contradiction.

• case $\nu_1 = 7$

In this case, $\sigma = 14, g_0 = 13^2 = 169$ and

$$\nu_1 + 1 = 8 = F(7) = 4x_1 + 6x_2.$$

Hence, $t_3 + t_6 = x_1 = 2, t_4 = t_5 = 0$ and so

$$t_2 + t_3 + t_6 + t_7 = 9, \quad t_2 + 3t_3 + 15t_6 + 21t_7 = 169.$$

From these, it follows that $2t_3 + 14t_6 + 20t_7 = 160; t_3 + 7t_6 + 10t_7 = 80$. Thus $6t_6 + 10t_7 = 80 - 2 = 78; 3t_6 + 5t_7 = 39$, a contradiction.

• case $\nu_1 = 6$

In this case $\sigma = 12, g_0 = 11^2 = 121$ and

$$\nu_1 + 1 = 7 = F(6) = 3x_1 + 4x_2.$$

Hence, $t_3 + t_5 = x_1 = 1, t_4 = x_2 = 1$ and

$$t_2 + t_3 + t_4 + t_5 + t_6 = 9, \quad t_2 + t_6 = 7, \quad t_2 + 3t_3 + 6t_4 + 10t_5 + 15t_6 = 121.$$

From these, it follows that

$$7 + 3 + 6 + 7t_5 + 14t_6 = 121, \quad 7t_5 + 14t_6 = 121 - 16 = 105, \quad t_5 + 2t_6 = 15.$$

Hence, $t_2 = t_3 = 0, t_4 = t_5 = 1, t_6 = 7$. Thus the type is $[12 * 12; 6^7, 5, 4]$ or its associates.

• case $\nu_1 = 5$

In this case $\sigma = 10, g_0 = 9^2 = 81$ and

$$\nu_1 + 1 = 6 = F(5) = 2x_1.$$

Hence, $t_3 + t_4 = x_1 = 3$. Thus

$$t_3 + t_4 + t_2 + t_5 = 9, 3t_3 + 6t_4 + t_2 + 10t_5 = 81,$$

$$9 + 3t_4 + 6 + 9t_5 = 81, 3t_4 + 9t_5 = 81 - 15 = 66, t_4 + 3t_5 = 22.$$

Since $t_4 \leq 3$ and $t_5 \leq 6$, it follows that $t_4 + 3t_5 \leq 21$; a contradiction.

• case $\nu_1 = 4$

In this case $\sigma = 8, g_0 = 7^2 = 49$ and

$$\nu_1 + 1 = 5 = F(4) = x_1.$$

Hence, $t_3 = x_1 = 5, t_2 + t_4 = 4$. Thus

$3t_3 + 6t_4 + t_2 = 49, 15 + 4 + 5t_4 = 49; 5t_4 = 30, t_4 = 6 > 4$; a contradiction.

• case $\nu_1 = 3$

In this case $\sigma = 6, g_0 = 5^2 = 25$ and $t_2 + t_3 = 9, \quad t_2 + 3t_3 = 25, \quad 2t_3 = 16$. Thus

$$t_2 = 1, \quad t_3 = 8.$$

Then $D^2 = 72 - 4 - 8 \cdot 9 = -4$, a contradiction.

16.2 case $\beta = 6$

Then $r = 10, K_S{}^2 = -2$ and so $\rho_{\nu_1} = 6\nu_1 - 4 - 6(\nu_1 - 1) = 2$. By Lemma 11

$$2 = \zeta_{\nu_1} + \theta_{\nu_1}.$$

By $\nu_1 \geq 3$, we get $3\nu_1 - 5 \geq 4$. Hence, if $p > 0$ then by Corollary, $2 = \rho_{\nu_1} \geq \tilde{A} + \gamma \geq 3\nu_1 - 5 \geq 4$, contradiction. Therefore, if $\theta_{\nu_1} > 0$, then $\sigma = 2\nu_1$ and $2 = \theta_{\nu_1} = \gamma = 2(\nu_1 - 2)(f + \nu_1 B - 2\nu_1)$. Hence, $\nu_1 = 3$ and $f + \nu_1 B - 2\nu_1 = 1$. Thus $B = 0, f = 7, \sigma = 6, g_0 = 30$. From

$$t_2 + t_3 = 10, t_2 + 3t_3 = 30,$$

it follows that $t_2 = 0, t_3 = 10$. The type is $[6 * 7; 3^{10}]$ or its associates.

If $\theta_{\nu_1} = 0$, then $\sigma = 2\nu_1$ and $f + \nu_1 B - 2\nu_1 = 0$. Hence, $2 = \zeta_{\nu_1}$.

$$\zeta_{\nu_1} = 2 = F(\nu_1) = (\nu_1 - 3)x_1 + 2(\nu_1 - 4)x_2 + \cdots .$$

Then from $2 = F(\nu_1) \geq \nu_1 - 3$, it follows that $\nu_1 \leq 5$.

• case $\nu_1 = 5, x_1 = 1$. Then $\sigma = f = 10, g_0 = 81, t_3 + t_4 = x_1 = 1$ and so

$$t_2 + t_5 + t_3 + t_4 = 10, \quad t_2 + 10t_5 + 3t_3 + 6t_4 = 81.$$

Hence,

$$9 + 9t_5 + 3 + 3t_4 = 81, 9t_5 + 3t_4 = 81 - 12 = 69, 3t_5 + t_4 = 23.$$

Since $t_4 = 0, 1$, there exist no solutions.

• case $\nu_1 = 4, x_1 = 2$. Then $\sigma = 8, t_3 = 2$. Hence, $g_0 = 49$. By genus formula,

$$t_2 + t_3 + t_4 = 10, \quad t_2 + 3t_3 + 6t_4 = 49.$$

Hence,

$$35t_4 = 35, \quad t_4 = 7, \quad t_2 = 1.$$

The type is $[8 * 8; 4^7, 3^2, 2]$ or its associates.

16.3 case $\beta = 7$

Then $r = 11, K_S{}^2 = -3$ and so $\rho_{\nu_1} = 3 - \nu_1$. Hence, $\nu_1 = 3$. Then $\sigma = 6, \rho_{\nu_1} = 0$ and so $\zeta_{\nu_1} = \theta_{\nu_1} = 0$. Then $3B + f = 6$, $g_0 = 25$ and

$$t_2 + t_3 = 11, \quad t_2 + 3t_3 = g_0 = 25.$$

There exists a solution to the effect that $t_2 = 4, t_3 = 7$ and so the type is $[6 * 6; 3^7, 2^4]$ or its associates.

16.4 case $\beta = 8$

Then $r = 12, {K_S}^2 = -4$ and so $\nu_1 = 2$. In this case, the type is $[4 * 5; 2^{12}]$ or its associates.

Theorem 10 *Suppose that $g = 0$ and $P_2[D] = 2$. Then $Z^2 = 0$ and*

1. *if $D^2 = -5$ then the type is either $[12 * 12; 6^7, 5, 4]$ or $[10 * 11; 5^9]$ or their associates.*
2. *If $D^2 = -6$ then the type is $[6 * 7; 3^{10}]$ or $[8 * 8; 4^7, 3^2, 2]$ or their associates.*
3. *If $D^2 = -7$ then the type is $[6 * 6; 3^7, 2^4]$ or its associates.*
4. *If $D^2 = -8$ then the type is $[4 * 5; 2^{12}]$ or its associates.*

16.5 curves parametrized by polynomials

Remark 3 *Rational curves C defined by parametrized $x = f(t) = t^n + a_1 t^{n-1} + \cdots + a_n, y = g(t) = t^m + b_1 t^{m-1} + \cdots + b_m, (n > m \geq 4, n \geq 6)$, where the a_j and the b_k are general, have $\sigma = m$ and Kodaira dimension 2, except for $(n, m) = (6, 5), (7, 4), (6, 4), (8, 4)$.*

The invariant D^2 is given by the following formula:
(1) $n = m - 1 \geq 6$. Then

$$D^2 = -n^2 + 6n - 4, \quad Z^2 = \frac{n^2 - 9n + 16}{2}.$$

(2) $n = mq_0 + r_0, 0 \leq r_0 < m, 2r_0 \leq m$ Then

$$D^2 = -(n-2)(m-2) + 2\delta(n, m) + q^*(n, m),$$
$$Z^2 = R(m, r_0) + 2(n-2)(m-2) - 2\delta(n, m).$$

(3) $n = mq_0 + r_0, 0 \leq r_0 < m, m = r_0 + r_1, r_1 < r_0$. Then

$$D^2 = -(n-2)(m-2) + 2\delta(n, m) + q^*(m, r_1).$$

[2]

[2] Note that the similar result was obtained by S.Usuda, independently.

17 rational curves with $Q = 1, 2$

While $\sigma \geq 4$, $Q = (2Z - D)^2 \geq 0$ has been established and so we shall investigate the type of pairs with small Q. By Proposition 11, if $Q = 0$ then $\sigma = 4$ and vice versa. By definition, $4Z^2 + 8 + D^2 = Q$. Hence, $Q - 8 + \beta = 4Z^2 \geq 0$; thus $\beta \geq 8 - Q$.

- When $\nu_1 \geq 3$, we get $(3Z - 2D) \cdot (2Z - D) \geq 0$. Hence,

$$(3Z - 2D) \cdot (2Z - D) = 3(Q + \beta - 8) + 28 - 4\beta \geq 0,$$

and so

$$3Q - \beta + 4 \geq 0.$$

Hence, $3Q + 4 \geq \beta$.

Suppose that $Q = 1$. Then $\beta = 7$ and $Z^2 = 0$. By Theorem 10, the type turns out to be $[6 * 6; 3^7, 2^4]$.

- When $\nu_1 \leq 2$, we obtain

$$1 = Q = (2Z - D)^2 = (\sigma - 4)(B\sigma + 2f - 8).$$

From this, it follows that $\sigma = 5, f = 7, g_0 = 14$ and the type is $[5 * 7, 1; 2^{14}]$, where $D^2 = -11, r = 14, K^2 = -6, Z^2 = -6 + 11 - 4 = 1$.

Suppose that $Q = 2$. If $\nu_1 \geq 3$, then $3 \cdot 2 - \beta + 4 \geq 0$, and so $10 - \beta \geq 0$. But from $4Z^2 + 8 - \beta = 2$, it follows that $\beta = 4Z^2 + 6 = 6$ or 10. So if $\beta = 6$, then $Z^2 = 0$. By Theorem 10, the type becomes $[6 * 7; 3^{10}]$ or $[8 * 8; 4^7, 3^2, 2]$ or their associates.

If $\beta = 10$, then $Z^2 = 1 = K^2 - D^2 - 4 = K^2 + 10 - 4, K^2 = -5$ and $r = 13$. Moreover, $\xi_0 = 12 - 13 = -1, \alpha = -4 + 10 = 6, \xi_2 = \sigma + f + \dfrac{B\sigma}{2} - (\xi_2 - 1)p$. It is not difficult to see that $p = 0$ and $\eta = 0$. Thus

$$\zeta = -2\nu_1 + 6 = F(\nu_1) \geq \nu_1 - 3.$$

Therefore, $\nu_1 = 3$ and so $\sigma = 6, g_0 = 25, t_2 + t_3 = 13, t_2 + 3t_3 = 25$. From this $t_2 = 7, t_3 = 6$. The type is $[6 * 6; 3^6, 2^7]$ or its associates , where $r = 13, D^2 = 72 - 54 - 28 = -10, Z^2 = 8 - 13 + 10 - 4 = 1$.

When $\nu_1 \leq 2$, we obtain $2 = Q = (\sigma - 4)(B\sigma + 2f - 8)$. Hence, it follows that $\sigma = 5, f = 5, g_0 = 16$ and the type is $[5 * 5; 2^{16}]$ or $[5 * 10, 2; 2^{16}]$, where $r = 16, D^2 = 50 - 64 = -14, Z^2 = 8 - 16 + 14 - 4 = 2$.

Combining Proposition 10 with the above argument, we establish the following result.

Theorem 11 *Assume that $Q = 1$. Then*

1. *the type is* $[6 * 6; 3^7, 2^\varepsilon]$, *where* $\varepsilon \leq 4$ *or*
2. *the type is* $[5 * 7, 1; 2^r]$ *or*
3. *the type is* $[7; 1]$ *or their associates, or*
4. *the type is* $[5; 1]$ *or*
5. *the type is* $[3 * 5, 1; 1]$.

Assume that $Q = 2$. Then

1. *the type is* $[8 * 8; 4^7, 3^2, 2^{\varepsilon'}]$, *where* $\varepsilon' \leq 1$ *or their associates or*
2. *the type is* $[6 * 7; 3^{10}]$ *or their associates, or*
3. *the type is* $[6 * 6; 2^\varepsilon, 3^6]$ *where* $g = 7 - \varepsilon$ *and* $D^2 = 4g - 10$ *or their associates, or*
4. *the type is* $[5 * 5; 2^r]$,
5. *the type is* $[5 * 10, 2; 2^r]$ *where* $g = 16 - r$ *and* $D^2 = 50 - 4r = 4g - 14$ *or*
6. *the type is* $[5 * 5; 2^r]$ *or* $[5 * 10, 2; 2^r]$ *or their associates or*
7. *the type is* $[3 * 3; 1]$.

Suppose that $Z^2 = 1$ and $g(D) = 0$. Then $Q = 4Z^2 + 8 + D^2 = 12 - \beta$. Hence, the next result follows immediately.

Corollary 6 *Suppose that $Z^2 = 1$ and $g(D) = 0$.*
If $\beta = -D^2 = 11$ then $Q = 1$ and thus the type is $[5 * 7, 1; 2^{14}]$.
If $\beta = 10$ then $Q = 2$ and thus the type is $[6 * 6; 3^6, 2^7]$ *or their associates.*

18 inequalities between Z^2 and D^2

For rational curves D, the following inequalities hold between Z^2 and D^2.

Proposition 17 *Suppose that $g = 0$ and $\kappa[D] = 2$. If $\nu_1 \leq 2$ and $\kappa[D] = 2$ then*

$$Z^2 = \frac{-(\sigma - 3)}{2(\sigma - 2)} D^2 + \frac{-\sigma^2 + 5\sigma - 8}{\sigma - 2},$$

$$P_2[D] = Z^2 + 2 = \frac{-(\sigma - 3)}{2(\sigma - 2)} D^2 + \frac{-\sigma^2 + 7\sigma - 12}{\sigma - 2}.$$

In particular, if $\sigma = 4$ then $Z^2 = \frac{-D^2}{4} - 2, P_2[D] = Z^2 + 2 = \frac{-D^2}{4}$.

If $\sigma = 5$ then $Z^2 = \frac{-D^2 - 8}{3}, P_2[D] = \frac{-D^2 - 2}{3}$.

Now we introduce the following regions

$$U_I = \{(x, y) \mid 4y > x - 8, 3y < x - 8\}$$

that is called vacant region I.

Define $R(\beta, Z^2) = \{(\beta, Z^2) \mid \text{for pairs } (S, D) \text{with rational } D\}$. Then from the previous result, we obtain

$$R(\beta, Z^2) \cap U_I = \emptyset.$$

18.1 curves parametrized by polynomials

Remark 4 *Rational curves C defined by parametrized $x = f(t) = t^n + a_1 t^{n-1} + \cdots + a_n, y = g(t) = t^m + b_1 t^{m-1} + \cdots + b_m, (n > m \geq 4, n \geq 6)$, where the a_j and the b_k are general, have $\sigma = m$ and Kodaira dimension 2, except for $(n, m) = (6, 5), (7, 4), (6, 4), (8, 4)$.*

The invariant D^2 is given by the following formula:
(1) $n = m - 1 \geq 6$. Then

$$D^2 = -n^2 + 6n - 4, \quad Z^2 = \frac{n^2 - 9n + 16}{2}.$$

(2) $n = mq_0 + r_0, 0 \leq r_0 < m, 2r_0 \leq m$ Then

$$D^2 = -(n - 2)(m - 2) + 2\delta(n, m) + q^*(n, m),$$
$$Z^2 = R(m, r_0) + 2(n - 2)(m - 2) - 2\delta(n, m).$$

(3) $n = mq_0 + r_0, 0 \leq r_0 < m, m = r_0 + r_1, r_1 < r_0$. Then

$$D^2 = -(n - 2)(m - 2) + 2\delta(n, m) + q^*(m, r_1).$$

3

[3] Note that the similar result was obtained by S.Usuda, independently.

$-D^2$	Z^2	n	m	$-D^2$	Z^2	n	m
4	-1	6	4	20	4	8	6
4	-1	7	4	20	4	8	7
4	-1	8	4	20	4	11	5
4	-1	6	5	22	5	9	6
8	0	9	4	24	4	17	4
11	1	7	5	26	6	12	5
11	1	7	6	28	5	18	4
12	1	10	4	28	5	19	4
12	1	11	4	28	5	20	4
12	1	12	4	28	7	10	6
14	2	8	5	28	7	11	6
14	2	9	5	28	7	12	6
14	2	10	5	29	7	13	5
16	2	13	4	29	7	14	5
20	3	14	4	29	7	15	5
20	3	15	4	31	8	9	7
20	3	16	4	31	8	9	8

Table 1: data of polynomial curves

18.2 curves parametrized by torus polynomials (*)

Here elements of $k[t, \frac{1}{t}]$ are said to be torus polynomials.

Let us consider rational curves C parametrized by torus polynomials

$$x = f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 + a_{-1}\frac{1}{t} + \cdots + a_{-n}\frac{1}{t^n},$$

$$y = g(t) = t^m + b_{-1}t^{m-1} + \cdots + b_0 + b_{-1}\frac{1}{t} + \cdots + b_{-m}\frac{1}{t^m},$$

where the a_j and the b_k are general. Under the assumption $n > m \geq 2$ have $\sigma = 2m$ and Kodaira dimension 2, except for $(n, m) = (6, 5), (7, 4), (6, 4), (8, 4)$.

19 curves with $P_2[D] = 3$

Next, the complete list of types of pairs (S, D) such that $P_2[D] = 3$ will be given. (The same result was obtained by S.Usuda independently at the same time.)

If $\kappa[D] = 2, g(D) = 0, P_2[D] = 3$, i.e., $Z^2 = 1$, then ${K_S}^2 - D^2 = 5$ and so $\beta = r - 3$. But since $8 - r = K_S^2 \leq -1$, we get $\beta = r - 3 \geq 9 - 3 = 6$. Furthermore, by definition

$$\rho_{\nu_1} = (\nu_1 Z - (\nu_1 - 1)D) \cdot (2Z - D) = 8\nu_1 - 4 - (\nu_1 - 1)\beta = (8 - \beta)\nu_1 + (\nu_1 - 1)\beta - 4.$$

First, we treat the case of curves with only double points.

Proposition 18 *If $g = 0, \nu_1 = 2, Z^2 = 1$ then the type is 1)* $[4 * 6; 2^{15}]$ *or its associates, where $D^2 = -12$, or 2)* $[5 * 7, 1; 2^{14}]$ *, where $D^2 = -11$.*

Proof: From $6 = (\sigma - 3)(B\sigma + 2f - 6)$, we have two cases 1) $\sigma - 3 = 1$ and 2) $\sigma - 3 = 2$.

case 1) $\sigma - 3 = 1$. Then $B\sigma + 2f - 6 = 6$ and so $3B + f = 6$. The type is $[4 * 6; 2^{15}]$ or its associates.

case 2) $\sigma - 3 = 2$. Then $B\sigma + 2f - 6 = 3$ and so $5B + 2f = 9$. Therefore, $B = 1, f = 2$ and the type is $[5 * 7, 1; 2^{14}]$. □

Second, we treat the case of curves with $\nu_1 = 3$.

Proposition 19 *If $g = 0, \nu_1 = 3, Z^2 = 1$ then*

$$\tau_5 = (\sigma - 5)(B\sigma + 2f - 10) = 2(14 - r).$$

Proof: Here, we shall prove the following formula:

If $\nu_1 = 3, K^2 = 8 - r$ and $\beta = -D^2$, then

$$\tau_5 = 10(g+4) + 4\beta - 6r.$$

In order to verify this, first consult genus formula and compute D^2:

$$t_2 + t_3 = r, t_2 + 3t_3 = g_0 - g, 4t_2 + 9t_3 = C^2 + \beta.$$

Then

$$2t_3 = g_0 - g - r, 5t_3 = C^2 + \beta - 4r,$$

and thus

$$5(g_0 - g - r) = 2(C^2 + \beta - 4r).$$

Since $2g_0 - 2 = Z_0 \cdot C$, it follows that

$$5Z_0 \cdot C + 10 - 4C^2 = 10g + 4\beta - 6r,$$

and by Formula I(Lemma 3)

$$\tau_5 = (5Z_0 - 4C) \cdot C + 50 = 10(g+4) + 4\beta - 6r.$$

□

Remark 5 *Replacing* β *by* $Z^2 + r - 4g - 4$, *we obtain*

$$\tau_5 = 24 + 4Z^2 - 6g - 2r.$$

Claim 7 $g_0 \le 3r$, *provided that* $\nu_1 = 3$ *and* $g = 0$.

Proof: The genus formula implies that

$$t_2 + t_3 = r, \quad t_2 + 3t_3 = g_0.$$

Then $g_0 - r = 2t_3 \le 2r$. Hence, $g_0 \le 3r$. □

The next result is easily verified.

Claim 8

$$B\sigma + 2f - 10 \ge \sigma - 5.$$

Since $t_3 > 0$, it follows that $\sigma \geq 6$ and hence, $r < 14$. But $r = \beta + 3 \geq 9$. Thus we have the following five cases:

(1) $r = 13$. Then $\tau_5 = (\sigma - 5)(B\sigma + 2f - 10) = 2$. Hence, $\sigma - 5 = 1, B\sigma + 2f - 10 = 2$. Thus $\sigma = 6$ and $3B + f = 6$, which implies $g_0 = 25$. By genus formula

$$t_2 + t_3 = 13, \quad t_2 + 3t_3 = 25.$$

Hence, $2t_3 = 12; t_3 = 6, t_2 = 7$. The type is $[6 * 6; 3^6, 2^7]$ or its associates.

(2) $r = 12$. Then $\tau_5 = (\sigma - 5)(B\sigma + 2f - 10) = 4$. Hence, a) $\sigma - 5 = 1, B\sigma + 2f - 10 = 4$; or b) $\sigma - 5 = 2, B\sigma + 2f - 10 = 2$.

In the case (2.a), we get $\sigma = 6$ and $3B + f = 7$, which implies $g_0 = 30$. By genus formula

$$t_2 + t_3 = 12, \quad t_2 + 3t_3 = 30.$$

Hence, $2t_3 = 18; t_3 = 9, t_2 = 3$. The type is $[6 * 7; 3^9, 2^3]$ or its associates.

In the case (2.b), we get $\sigma = 7$ and $7B + 2f = 12$, a contradiction.

(3) $r = 11$. Then $\tau_5 = (\sigma - 5)(B\sigma + 2f - 10) = 6$. Hence, a) $\sigma - 5 = 1, B\sigma + 2f - 10 = 6$; or b) $\sigma - 5 = 2, B\sigma + 2f - 10 = 3$.

In the case (3.a), we get $\sigma = 6$ and $3B + f = 8$

, which implies $g_0 = 35$. By genus formula

$$t_2 + t_3 = 11, \quad t_2 + 3t_3 = 35.$$

Hence, $2t_3 = 24; t_3 = 12$; a contradiction.

In the case (3.b), we get $\sigma = 7$ and $7B + 2f = 13$, which implies $B = 1, f = 3, g_0 = 33$. By genus formula

$$t_2 + t_3 = 11, \quad t_2 + 3t_3 = 33.$$

Hence, $2t_3 = 22; t_3 = 11, t_2 = 0$. The type is $[7 * 10, 1; 3^{11}]$.

(4) $r = 10$. Then $g_0 \leq 3r = 30$ and moreover, $\tau_5 = (\sigma - 5)(B\sigma + 2f - 10) = 8$. Hence, a) $\sigma - 5 = 1, B\sigma + 2f - 10 = 8$; or b) $\sigma - 5 = 2, B\sigma + 2f - 10 = 4$.

In the case (4.a), we get $\sigma = 6$ and $3B + f = 9$, which implies $g_0 = 40$; a contradiction.

In the case (4.b), we get $\sigma = 7$ and $7B + 2f = 14$, which implies $g_0 = 36$; a contradiction.

(5) $r = 9$. Then $g_0 \le 3r = 27$ and moreover, $\tau_5 = (\sigma - 5)(B\sigma + 2f - 10) = 10$. Hence, a) $\sigma - 5 = 1, B\sigma + 2f - 10 = 10$; or b) $\sigma - 5 = 2, B\sigma + 2f - 10 = 5$.

In the case (5.a), we get $\sigma = 6$ and $3B + f = 10$, which implies $g_0 = 45$, a contradiction.

In the case (5.b), we get $\sigma = 7$ and $7B + 2f = 15$, which implies $B = 1, f = 4$. Hence, $g_0 = 18 + 21 = 39$; a contradiction.

Thus the following result has been established.

Proposition 20 *Suppose that* $g = 0, \nu_1 = 3$ *and* $Z^2 = 1$. *Then* $r = 11, 12, 13$ *and*

1. *if* $r = 13$ *then the type is* $[6 * 6; 3^6, 2^7]$ *or its associates.*
2. *If* $r = 12$ *then the type is* $[6 * 7; 3^9, 2^3]$ *or its associates.*
3. *If* $r = 11$ *then the type is* $[7 * 10, 1; 3^{11}]$.

Owing to the previous result, we may suppose that $\nu_1 \ge 4$.

Letting w be $(2Z - D)^2 \ge 0$, we get

$$w = (2Z - D)^2 = 4Z^2 - 4Z \cdot D + D^2 = 12 - \beta.$$

Since in the cases $w = 0, 1, 2$, all types have been already enumerated in Proposition 10, we may assume $w \ge 3$. Thus we have the following four cases to examine, separately:

(1) $\beta = 6$, (2) $\beta = 7$, (3) $\beta = 8$, (4) $\beta = 9$.

19.1 case $\beta = 6$

Then $\rho_{\nu_1} = 2\nu_1 + 2$; hence, by Lemma 11,

$$2\nu_1 + 2 = \zeta_{\nu_1} + \theta_{\nu_1}.$$

First assume that $\theta_{\nu_1} = 0$.

Then $\sigma = 2\nu_1$ and $f + B\nu_1 - 2\nu_1 = 0$ which implies that $g_0 = (2\nu_1 - 1)(f - 1) + B\nu_1(2\nu_1 - 1) = (2\nu_1 - 1)^2$.

$$2\nu_1 + 2 = F(\nu_1) = (\nu_1 - 3)x_1 + 2(\nu_1 - 4)x_2 + 3(\nu_1 - 5)x_3 + \cdots .$$

To find the maximal ν_1, we suppose that $2\nu_1 + 2 \ge 3(\nu_1 - 5)$. Then $\nu_1 \ge 17$.

• case $\nu_1 = 17$. From hypothesis, $x_3 = 1$ and $x_j = 0$ if $j \neq 3$. Therefore, $\sigma = 34$ and $g_0 = 33^2 = 1089$, and moreover,

$$1 = x_3 = t_5 + t_{14}, t_2 + t_{17} + t_5 + t_{14} = 9, t_2 + 136t_{17} + 10t_5 + 91t_{14} = 9,$$

$$135t_{17} + 9t_5 + 90t_{14} = 1080.$$

Therefore,

$$135t_{17} + 81t_{14} = 1080 - 9 = 1071.$$

There exist no solutions.

While $\nu_1 \leq 16$, we get $2\nu_1 + 2 \geq (\nu_1 - 3) + 2(\nu_1 - 4)$, which implies $\nu_1 \geq 13$.

• case $\nu_1 = 13$. From

$$28 = F(13) = 10x_1 + 18x_2 + 24x_3 + 28x_4.$$

Then we get two solutions 1) $x_1 = x_2 = 1, x_j = 0$ and 2) $x_1 = x_2 = x_3 = 0, x_4 = 1$. Therefore, since $\sigma = 26, g_0 = 25^2 = 625$, it follows that in the case 1)

$$1 = x_1 = t_3 + t_{12} = x_2 = t_4 + t_{11},$$

$$t_2 + t_{13} + t_3 + t_{12} + t_4 + t_{11} = 9, \quad t_2 + 78t_{13} + 3t_3 + 66t_{12} + 6t_4 + 55t_{11} = 625,$$

$$77t_{13} + 2t_3 + 65t_{12} + 5t_4 + 54t_{11} = 616,$$

$$77t_{13} + 2 + 63t_{12} + 5 + 49t_{11} = 616,$$

$$77t_{13} + 63t_{12} + 49t_{11} = 609.$$

$$11t_{13} + 9t_{12} + 7t_{11} = 87.$$

The equations have no solutions.

In the case 2),

$$1 = x_4 = t_6 + t_9, \quad t_2 + t_6 + t_9 + t_{13} = 9, \quad t_2 + 78t_{13} + 15t_6 + 36t_9 = 625.$$

Hence,

$$77t_{13} + 14t_6 + 35t_9 = 616, \quad 11t_{13} + 2t_6 + 5t_9 = 88.$$

Thus,

$$11t_{13} + 3t_9 = 86; \quad t_9 = -1.$$

The equations have no solutions.

• case $\nu_1 = 12$. Then

$$26 = F(12) = 9x_1 + 16x_2 + 21x_3 + 24x_4 + 25x_5 + \cdots .$$

The equation has no solutions.
• case $\nu_1 = 11$. Then $\sigma = 22, f = 2\nu_1 - B\nu_1 = 22 - 11B$ and $g_0 = 21^2 = 441$. The equation

$$24 = F(11) = 8x_1 + 14x_2 + 18x_3 + 20x_4$$

has a solution $x_1 = 3, x_2 = x_3 = x_4 = 0$. Hence, $t_3 + t_{10} = x_1 = 3$ and so

$$t_2 + t_3 + t_{10} + t_{11} = 9, \quad t_2 + 3t_3 + 45t_{10} + 55t_{11} = 441,$$

$$2t_3 + 44t_{10} + 54t_{11} = 6 + 42t_{10} + 54t_{11} = 432.$$

Thus,

$$42t_{10} + 54t_{11} = 426,$$

and so

$$7t_{10} + 9t_{11} = 71.$$

But the equation has no solutions.

• case $\nu_1 = 10$. Then $\sigma = 20, f = 2\nu_1 - B\nu_1 = 20 - 10B$ and $g_0 = 19^2 = 361$. The equation

$$22 = F(10) = 7x_1 + 12x_2 + 15x_3 + 16x_4$$

has a solution $x_1 = 1, x_2 = x_4 = 0, x_3 = 1$. Hence, $t_3 + t_9 = t_5 + t_7 = 1$ and so

$$t_2 + t_3 + t_9 = t_5 + t_7 + t_{10} = 9, \quad t_2 + 3t_3 + 36t_9 + 10t_5 + 21t_7 + 45t_{10} = 361,$$

$$2t_3 + 35t_9 + 9t_5 + 20t_7 + 44t_{10} = 352.$$

Thus,

$$2 + 33t_9 + 9 + 11t_7 + 44t_{10} = 352,$$

and so

$$33t_9 + 11t_7 + 44t_{10} = 341.$$

Hence,

$$3t_9 + t_7 + 4t_{10} = 31.$$

The equation has a solution $t_9 = 1, t_7 = 0, t_{10} = 7, t_5 = 1$. The type is $[20 * 20; 10^7, 9, 5]$ or its associates.

• case $\nu_1 = 9$. Then $\sigma = 18, f = 2\nu_1 - B\nu_1 = 18 - 9B$ and $g_0 = 17^2 = 289$. The equation

$$20 = F(9) = 6x_1 + 10x_2 + 12x_3$$

has a solution $x_2 = 2, x_1 = x_3 = 0$ and hence,

$$t_4 + t_7 = 2, \quad t_2 + t_9 + t_4 + t_7 = 9, \quad t_2 + 36t_9 + 6t_4 + 21t_7 = 289,$$

$$35t_9 + 5t_4 + 20t_7 = 280, \quad 35t_9 + 15t_7 = 270.$$

Hence, $7t_9 + 3t_7 = 54$, and then $t_9 = 6, t_7 = 4$; a contradiction.

• case $\nu_1 = 8$. Then $\sigma = 16, f = 2\nu_1 - B\nu_1 = 16 - 8B$ and $g_0 = 15^2 = 225$. The equation

$$18 = F(8) = 5x_1 + 8x_2 + 9x_3$$

has two solutions 1) $x_3 = 2$, and 2) $x_1 = 2, x_2 = 1$.

In case 1), $t_5 = x_3 = 2$ and

$$t_2 + t_8 + t_5 = 9, \quad t_2 + 28t_8 + 10t_5 = 225.$$

Hence,

$$27t_8 + 9t_5 = 216; \quad 3t_8 + t_5 = 24.$$

Then $3t_8 = 24 - 2 = 22$, a contradiction.

In case 2), $t_3 + t_7 = 2, t_4 + t_6 = 1$ and so

$$t_2 + t_8 + t_3 + t_7 + t_4 + t_6 = 9, \quad t_2 + 28t_8 + 3t_3 + 21t_7 + 6t_4 + 15t_6 = 225.$$

Hence,

$$27t_8 + 2t_3 + 20t_7 + 5t_4 + 14t_6 = 216 \quad 27t_8 + 4 + 18t_7 + 5 + 9t_6 = 216.$$

Then $27t_8 + 18t_7 + 9t_6 = 207$; hence, $3t_8 + 2t_7 + t_6 = 23$. There exists one solution $t_8 = 6, t_7 = 2, t_6 = 1$. The type is $[16 * 16; 8^6, 7^2, 6]$ or its associates.

• case $\nu_1 = 7$. Then $\sigma = 14, f = 2\nu_1 - B\nu_1 = 14 - 7B$ and $g_0 = 13^2 = 169$. The equation

$$16 = F(7) = 4x_1 + 6x_2$$

has two solutions : 1) $x_1 = 4$ and 2) $x_1 = 1, x_2 = 2$.

In case 1), $t_3 + t_6 = x_1 = 4$ and

$$t_2 + t_7 + t_3 + t_6 = 9, \quad t_2 + 21t_7 + 3t_3 + 15t_6 = 169.$$

Hence, $20t_7 + 2t_3 + 14t_6 = 160$.

$$20t_7 + 12t_6 = 152; \quad 5t_7 + 3t_6 = 38.$$

Then $t_7 = 7, t_6 = 1, t_3 = 3$; a contradiction.

In case 2), $t_3 + t_6 = x_1 = 1, t_4 + t_5 = x_2 = 2$ and

$$t_2 + t_7 + t_3 + t_6 + t_4 + t_5 = 9, \quad t_2 + 21t_7 + 3t_3 + 15t_6 + 6t_4 + 10t_5 = 169.$$

Hence, $20t_7 + 2t_3 + 14t_6 + 5t_4 + 9t_5 = 160$, $20t_7 + 2 + 12t_6 + 10 + 4t_5 = 160$ and so $20t_7 + 12t_6 + 4t_5 = 148$; thus $5t_7 + 3t_6 + t_5 = 37$. There exists no solution.

• case $\nu_1 = 6$. Then $\sigma = 12, f = 2\nu_1 - B\nu_1 = 12 - 6B$ and $g_0 = 11^2 = 121$. The equation

$$14 = F(6) = 3x_1 + 4x_2$$

has one solution $x_1 = x_2 = 2$ and hence $t_3 + t_5 = x_1 = 2, t_4 = x_2 = 2$ and so

$$t_2 + t_6 + t_3 + t_5 + t_4 = 9, \quad t_2 + 15t_6 + 3t_3 + 10t_5 + 6t_4 = 121.$$

Hence,

$$14t_6 + 2t_3 + 9t_5 + 5t_4 = 112, \quad 14t_6 + 4 + 7t_5 + 10 = 112.$$

Therefore, $2t_6 + t_5 = 14$; hence, $t_6 = 7, t_5 = 0, t_3 + t_5 = 2, t_4 = 2, t_3 = 2, r > 9$; a contradiction.

• case $\nu_1 = 5$. Then $\sigma = 10, f = 2\nu_1 - B\nu_1 = 10 - 5B$ and $g_0 = 81$. The equation $12 = F(5) = 2x_1$ has a solution $x_1 = 6$ and so $t_3 + t_4 = x_1 = 6$. Hence,

$$t_2 + t_5 + t_3 + t_4 = 9; \quad t_2 + 10t_5 + 3t_3 + 6t_4 = 81.$$

Therefore,

$$t_2 + t_5 = 3; \quad 3 + 9t_5 + 3 \cdot 6 + 3t_4 = 81.$$

Hence, $9t_5 + 3t_4 = 60; 3t_5 + t_4 = 20$. But $3t_5 + t_4 \leq 9 + 6 = 15 < 20$; a contradiction.

• case $\nu_1 = 4$. If $\nu_1 = 4$ then $10 = F(4) = x_1 = t_3 > 9$; a contradiction.

19.1.1 case $p > 0$.

Second, assume that $p = \sigma - 2\nu_1 > 0$. Then $\tilde{A} = (p+3\nu_1-2)B+2f-2\nu_1-4$ and assume $\nu_1 \geq 4$.

We shall study in the following cases : 1) $B = 0$, 2) $B = 1$, 3) $B \geq 2$ separately.

case 1) $B = 0$. Then $\tilde{A} = 2f - 2\nu_1 - 4, f = \sigma + u = p + 2\nu_1 + u, \gamma = 2(\nu_1 - 2)(u+p)$, , where $p > 0, u \geq 0$.

From

$$2\nu_1 + 2 = \zeta + (2p + 2u + 2\nu_1 - 4)p + 2(\nu_1 - 2)(u + p),$$

it follows that

$$\zeta = 2(1 - u - 2p)\nu_1 + 8p + 4u - 4p^2 - 4pu + 2.$$

By $\nu_1 \geq 4$,

$$\zeta \leq 8(1 - u - 2p)\nu_1 + 8p + 4u - 4p^2 - 4pu + 2 = 10 - 8p - 2p^2 - 2pu - 4u.$$

Hence, $p = 1, u = 0, \zeta = 0, \nu_1 = 4$; thus $\sigma = 9, g_0 = 64, t_3 = 0$. By genus formula

$$t_2 + t_3 + t_4 = 9, \quad t_2 + 3t_3 + 6t_4 = 64.$$

Thus $5t_4 = 55, t_4 = 11$; a contradiction.

case 2) $B = 1$. Then $\tilde{A} = p + 3\nu_1 - 2 + 2u - 4 = p + 3\nu_1 + 2u - 6, f = \nu_1 + u, \gamma = 2(\nu_1 - 2)u$, where $p > 0, u \geq 0$. Thus

$$\zeta_{\nu_1} = (2 - 3p - 2u)\nu_1 + 2 + 4u + (6 - p - 2u)p.$$

By $\nu_1 \geq 4$,

$$0 \leq \zeta_{\nu_1} \leq 8 - 12p - 8u + 2 + 4u + (6 - p - 2u)p = 10 - 6p - 4u - (p + 2u)p.$$

Therefore, $p = 1, u = 0$.

Hence, $\zeta_{\nu_1} = 7 - \nu_1$. Recalling the definition of ζ_{ν_1}, we obtain

$$\zeta_{\nu_1} = 7 - \nu_1 = F(\nu_1) = (\nu_1 - 3)x_1 + \cdots .$$

- If $\zeta_{\nu_1} = 0$ then $\nu_1 = 7, t_3 = \cdots = t_6 = 0$ and so $\sigma = 15, f = 7$. Therefore, $g_0 = 14 \cdot 6 + 7 \cdot 15 = 189$. By genus formula,

$$t_2 + t_7 = 9, \quad t_2 + 21t_7 = 189.$$

From this, it follows that $t_2 = 0, t_7 = 9$. Hence, the type is $[15 * 22, 1; 7^9]$.

• If $\zeta_{\nu_1} > 0$ then $7 - \nu_1 = F(\nu_1) \geq (\nu_1 - 3)$. Hence, $\nu_1 \leq 5$. Thus here are two cases:

case (1) $\nu_1 = 5$. Then $\sigma = 11, f = 5, g_0 = 95$. Since $7 - \nu_1 = 2 = F(5) = 2x_1$, it follows that $x_1 = t_3 + t_4 = 1$. By genus formula

$$t_2 + t_5 + t_3 + t_4 = 9, \quad t_2 + 10t_5 + 3t_3 + 6t_4 = 95.$$

This yields $t_4 + 3t_5 = 28$ and so $t_5 = 9, t_4 = 1, r \geq 10$; a contradiction.

case (2) $\nu_1 = 4$. Then $\sigma = 9, f = 4, g_0 = 60$. Since $7 - \nu_1 = 3 = F(4) = x_1$, it follows that $x_1 = t_3 = 3$. By genus formula

$$t_2 + t_3 + t_4 = 9, \quad t_2 + 3t_3 + 6t_4 = 60.$$

Hence, $2t_3 + 5t_4 = 51; 5t_4 = 45, t_4 = 9, r > 9 + 3 = 12$, which is a contradiction.

case 3) $B \geq 2$. Then we can derive a contradiction by the same argument as before.

19.1.2 case $p = 0$.

Third, assume that $\sigma = 2\nu_1$. Then $\gamma = 2(\nu_1 - 2)(f + \nu_1 B - 2\nu_1) > 0$.

We shall study in the following cases : 1) $B = 0$, 2) $B = 1$, 3) $B \geq 2$, separately.

case 1) $B = 0$: $f = \sigma + u = 2\nu_1 + u$ and $\gamma = 2(\nu_1 - 2)u > 0$. Hence, $2\nu_1 + 2 = \zeta + 2(\nu_1 - 2)u$, i.e., $\zeta = 2(1 - u)\nu_1 + 2 + 4u$.

In the case when $u = 1$, we get $\zeta = 6$. Thus $6 = F(\nu_1) \geq \nu_1 - 3$. Hence, $\nu_1 \leq 9$. We shall examine the following 7 cases separately.

• case $\nu_1 = 9$.

Then $6 = F(9) = (9-3)x_1$ and so $t_3 + t_8 = x_1 = 1$. Further, $\sigma = 18, f = 19, g_0 = 17 \cdot 18 = 306$. By genus formula,

$$t_3 + t_8 + t_2 + t_9 = 9, \quad 3t_3 + 28t_8 + t_2 + 36t_9 = 306,$$

$$2t_3 + 27t_8 + 35t_9 = 297, \quad 25t_8 + 35t_9 = 295.$$

Hence, $5t_8 + 7t_9 = 59$; a contradiction.

• case $\nu_1 = 8$.

Then $6 = F(8) = 5x_1 + 8x_2 + 9x_3 + \cdots$, which has no solutions.

• case $\nu_1 = 7$.

Then $6 = F(7) = 4x_1 + 6x_2$, which has a solution $t_3 + t_6 = x_1 = 0, t_4 + t_5 = x_2 = 1$.

Further, $\sigma = 14, f = 15, g_0 = 13 \cdot 14 = 182$. By genus formula,

$$t_4 + t_5 + t_2 + t_7 = 9, \quad 6t_4 + 10t_5 + t_2 + 21t_7 = 182,$$

$$5t_4 + 9t_5 + 20t_7 = 173, \quad 4t_5 + 20t_7 = 173 - 5 = 168.$$

Hence, $t_5 + 5t_7 = 42$; a contradiction.

• case $\nu_1 = 6$.

Then $6 = F(7) = 3x_1 + 4x_2$, which has a solution $t_3 + t_5 = x_1 = 2, t_4 = x_2 = 0$.

Further, $\sigma = 12, f = 13, g_0 = 11 \cdot 12 = 132$. By genus formula,

$$t_3 + t_5 + t_2 + t_6 = 9, \quad 3t_3 + 10t_5 + t_2 + 15t_6 = 132,$$
$$2t_3 + 9t_5 + 14t_6 = 123, \quad 7t_5 + 14t_6 = 119,$$

Hence, $t_5 + 2t_6 = 17$; a contradiction.

•

case $\nu_1 = 5$.

Then $6 = F(5) = 2x_1$, which has a solution $t_3 + t_4 = x_1 = 3$. Further, $\sigma = 10, f = 11, g_0 = 9 \cdot 10 = 90$. By genus formula,

$$t_3 + t_4 + t_2 + t_5 = 9, \quad 3t_3 + 6t_4 + t_2 + 10t_5 = 90,$$

$$2t_3 + 5t_4 + 9t_5 = 81, \quad 3t_4 + 9t_5 = 75.$$

Hence, $t_4 + 3t_5 = 25,$; a contradiction.

• case $\nu_1 = 4$.

Then $6 = F(4) = x_1$, which has a solution $t_3 = x_1 = 6$. Further, $\sigma = 8, f = 9, g_0 = 7 \cdot 8 = 56$. By genus formula,

$$t_3 + t_2 + t_4 = 9, \quad t_2 + t_4 = 3, \quad 3t_3 + t_2 + 6t_4 = 56.$$

Hence, $2t_3 + 5t_4 = 47, \quad 5t_4 = 47 - 12 = 35, \quad t_4 = 7$; a contradiction.

In the case when $u = 2$, $\zeta = 10 - 2\nu_1$. Thus $10 - 2\nu_1 = F(\nu_1) \geq \nu_1 - 3$. Hence, $\nu_1 \leq 4$ and so$\nu_1 = 4$. Therefore, $2 = F(4) = x_1 = t_3$. Further, $\sigma = 8, f = 10, g_0 = 7 \cdot 9 = 63$.

By genus formula,

$$t_3 + t_2 + t_4 = 9, \quad t_2 + t_4 = 7, \quad 3t_3 + t_2 + 6t_4 = 63.$$

$2t_3 + 5t_4 = 54, \quad 5t_4 = 54 - 4 = 50, \quad t_4 = 8$, a contradiction.

In the case when $u = 3$, $2\nu_1 + 2 = \zeta = 14 - 4\nu_1$. Hence, $\nu_1 = 2$.

case 2) $B = 1$: $f = \nu_1 + u$ and $\gamma = 2(\nu_1 - 2)u > 0$. Hence, $2\nu_1 + 2 = \zeta + 2(\nu_1 - 2)u$, i.e., $\zeta = 2(1-u)\nu_1 + 2 + 4u$.

case 3) $B \geq 2$: $f = u$ and $\gamma = 2(\nu_1 - 2)(f + \nu_1 B - 2\nu_1) > 0$. When $B = 2$, $2\nu_1 + 2 = \zeta + 2(\nu_1 - 2)u$, i.e., $\zeta = 2(1-u)\nu_1 + 2 + 4u$.

In both cases, we are able to derive contradictions.

19.2 case $\beta = 7$

Then $\rho_{\nu_1} = \nu_1 + 3$; Hence, by Lemma 11,

$$\nu_1 + 3 = \zeta_{\nu_1} + \theta_{\nu_1}.$$

First assume that $\theta_{\nu_1} = 0$.

Then $\sigma = 2\nu_1$ and $f + B\nu_1 - 2\nu_1 = 0$.

$$\nu_1 + 3 = F(\nu_1) = (\nu_1 - 3)x_1 + 2(\nu_1 - 4)x_2 + 3(\nu_1 - 5)x_3 + \cdots.$$

To find the maximal ν_1, we suppose that $\nu_1 + 3 \geq 2x_2(\nu_1 - 4)$. Then $\nu_1 \geq 11$.

• case $\nu_1 = 11$. From hypothesis, $x_2 = 1$ and $x_j = 0$ if $j \neq 2$. Therefore, since $\sigma = 22, g_0 = 21^2 = 441$,

$$1 = x_2 = t_4 + t_9, t_4 + t_9 + t_2 + t_{11} = 10, 6t_4 + 36t_9 + t_2 + 55t_{11} = 441.$$

$$5t_4 + 35t_9 + 54t_{11} = 431; 5 + 30t_9 + 54t_{11} = 431.$$

Therefore, $30t_9 + 54t_{11} = 426, 5t_9 + 9t_{11} = 71$. There exist no solutions.

• case $\nu_1 = 10$.

$$13 = F(10) = 7x_1 + 12x_2.$$

There exist no solutions.

• case $\nu_1 = 9$ Then $\sigma = 18, g_0 = 17^2 = 289$.

$$12 = F(9) = 6x_1 + 10x_2 + 12x_3.$$

There exist two solutions (1) $t_3 + t_8 = x_1 = 2$ and (2) $t_5 + t_6 = x_3 = 1$.

In case (1),

$$t_3 + t_8 = 2, t_3 + t_8 + t_2 + t_9 = 10, 3t_3 + 28t_8 + t_2 + 36t_9 = 289.$$

Hence, $2t_3 + 27t_8 + 35t_9 = 279; \quad 5t_8 + 7t_9 = 55$, a contradiction.

In case (2),

$$t_5 + t_6 = 1, t_5 + t_6 + t_2 + t_9 = 10, 10t_5 + 15t_6 + t_2 + 36t_9 = 289.$$

Hence, ; $\quad t_6 + 7t_9 = 54$, a contradiction.

• case $\nu_1 = 8$.

$$11 = F(8) = 5x_1 + 8x_2 + 9x_3.$$

There exist no solutions.

• case $\nu_1 = 7$ Then $\sigma = 14, g_0 = 13^2 = 169$ and hence we get

$$10 = F(7) = 4x_1 + 6x_2.$$

Then $x_1 = x_2 = 1$. Hence, $t_3 + t_6 = x_1 = t_4 + t_5 = x_2 = 1$. Thus

$$t_3 + t_6 + t_4 + t_5 + t_2 + t_7 = 10, \quad 3t_3 + 15t_6 + 6t_4 + 10t_5 + t_2 + 21t_7 = 169.$$

Hence, $2t_3 + 14t_6 + 5t_4 + 9t_5 + 20t_7 = 159, 12t_6 + 4t_5 + 20t_7 = 152$.

Therefore, $3t_6 + t_5 + 5t_7 = 38$. Then $t_6 = 1, t_5 = 0, t_7 = 7$ and the type is $[14 * 14; 7^7, 6, 4, 2]$.

• case $\nu_1 = 6$ Then $\sigma = 12, g_0 = 11^2 = 121$ and hence we get

$$9 = F(6) = 3x_1 + 4x_2.$$

Then $x_1 = 3$. Hence, $t_3 + t_5 = x_1 = 3$. Thus

$$t_3 + t_5 + t_2 + t_6 = 10, \quad 3t_3 + 10t_5 + t_2 + 15t_6 = 121.$$

Hence, $2t_3 + 9t_5 + 14t_6 = 111, 2 + 7t_5 + 14t_6 = 111; t_5 + 2t_6 = 15$.

Therefore, we have two cases (1) $t_5 = 1, t_6 = 7$ and (2) $t_5 = 3, t_6 = 6$. In case (1), the type is $[12 * 12; 6^7, 5, 3^2]$ or its associates and in case (2), the type is $[12 * 12; 6^6, 5^3, 2]$ or its associates.

• case $\nu_1 = 5$ Then $\sigma = 10, g_0 = 9^2 = 81$ and hence we get

$$8 = F(5) = 2x_1.$$

Then $x_1 = 4$. Hence, $t_3 + t_4 = x_1 = 4$. Thus

$$t_3 + t_4 + t_2 + t_5 = 10, \quad 3t_3 + 6t_4 + t_2 + 10t_5 = 81.$$

Hence,

$$2t_3 + 5t_4 + 9t_5 = 71; 3t_4 + 9t_5 = 71 - 8 = 63.$$

Therefore, $t_4 + 3t_5 = 21$; Hence, $t_5 = 6, t_4 = 3, t_3 = 1$ and the type is $[10 * 10; 5^6, 4^3, 3]$ or its associates.

• case $\nu_1 = 4$ Then $\sigma = 8, g_0 = 7^2 = 49$ and hence we get

$$7 = F(3) = t_3.$$

Thus

$$t_3 + t_2 + t_4 = 10, \quad 3t_3 + t_2 + 6t_4 = 49.$$

Therefore, $t_2 + t_4 = 3, 21 + 3 + 5t_4 = 49; 5t_4 = 25; t_4 = 5$; a contradiction.

19.2.1 case $\theta_{\nu_1} > 0$

Next assume that $\theta_{\nu_1} > 0, p > 0$.

Then $\nu_1 + 3 = \zeta + \theta \geq 3\nu_1 - 5$; hence, $\nu_1 \leq 4$.

When $\nu_1 = 4$, $B = 1, \sigma = 9$; hence, $\zeta = 0, f = 4$. Thus $g_0 = 8 \cdot 3 + 36 = 60$. Therefore,

$$t_2 + t_4 = 10, \quad t_2 + 6t_4 = 60.$$

Hence, $5t_4 = 50$ and so $t_4 = 10, t_2 = 0$. The type is $[9 * 13, 1; 4^{10}]$.

Finally, assume that $\theta_{\nu_1} > 0, p = 0$. Then $\tilde{A} = 0$ and $\nu_1 + 3 = \zeta + \theta_{\nu_1}, \theta_{\nu_1} = \gamma = 2(\nu_1 - 2)(f + \nu_1 B - 2\nu_1) > 0$.

We shall study in the following cases : 1) $B = 0$, 2) $B = 1$ and 3) $B \geq 2$, separately.

case 1) $B = 0$: $f = \sigma + u = 2\nu_1 + u, \gamma = 2(\nu_1 - 2)(f - 2\nu_1) = 2(\nu_1 - 2)u$.

case 2) $B = 1$: $f = \nu_1 + u, \gamma = 2(\nu_1 - 2)(f - \nu_1) = 2(\nu_1 - 2)u$.

case 3) $B = 2$: $f = u, \gamma = 2(\nu_1 - 2)f = 2(\nu_1 - 2)u$.

In any cases, $\nu_1 + 3 = \zeta + \theta_{\nu_1} \geq 2(\nu_1 - 2)u$ and thus $3 + 4u \geq (2u - 1)\nu_1$.

If $u = 1$ then $\nu_1 \leq 7$.

• case $\nu_1 = 7$ Then $\sigma = 14, g_0 = 13 \cdot 14 = 182$. $\zeta = \nu_1 + 3 - \theta_{\nu_1} = \nu_1 + 3 - 2(\nu_1 - 2) = 7 - \nu_1 = 0$. Hence, $t_3 = t_4 = t_5 = t_6 = 0$. By genus formula,

$$t_2 + t_7 = 10, \quad t_2 + 21t_7 = 182, \quad 20t_7 = 172.$$

• case $\nu_1 = 6$

$$\zeta = \nu_1 + 3 - \theta_{\nu_1} = \nu_1 + 3 - 2(\nu_1 - 2) = 7 - \nu_1 = 1.$$

$1 = \zeta = 3x_1 + \cdots$, a contradiction.

• case $\nu_1 = 5$ Then $\sigma = 10, g_0 = 9 \cdot 10 = 90$ and hence we get

$$\zeta = \nu_1 + 3 - \theta_{\nu_1} = \nu_1 + 3 - 2(\nu_1 - 2) = 7 - \nu_1 = 2.$$

$2 = \zeta = 2x_1 + \cdots$. Hence, $t_3 + t_4 = x_1 = 1$.

$$t_3 + t_4 + t_2 + t_5 = 10, \quad 3t_3 + 6t_4 + t_2 + 10t_5 = 90,$$

$$2t_3 + 5t_4 + 9t_5 = 80; \quad 2 + 3t_4 + 9t_5 = 80.$$

Therefore, $t_4 + 3t_5 = 26$; hence, $t_4 = 2, t_5 = 8, t_3 = -1$; a contradiction.

• case $\nu_1 = 4$ Then $\sigma = 8, g_0 = 7 \cdot 8 = 56$ and hence we get

$$\zeta = \nu_1 + 3 - \theta_{\nu_1} = \nu_1 + 3 - 2(\nu_1 - 2) = 7 - \nu_1 = 3.$$

Hence, $3 = \zeta = t_3$ and so

$$t_3 + t_4 + t_2 = 10, \quad 3t_3 + 6t_4 + t_2 = 56,$$

$5t_4 = 40; t_4 = 8 > 7$; a contradiction.

case 4) $B \geq 3$: $f = u, \gamma = 2(\nu_1 - 2)(f + B\nu_1 - 2\nu_1) \geq 2(\nu_1 - 2)(u + \nu_1)$. Hence,

$$\zeta = \nu_1 + 3 - \theta_{\nu_1} \leq \nu_1 + 3 - 2(\nu_1 - 2)(u + \nu_1) \leq \nu_1(1 - 4 - 2\nu_1) + 3 - 2(\nu_1 - 2) < 0;$$

a contradiction.

When $u > 1$, by the similar way, we can derive a contradiction.

19.3 case $\beta = 8$

Then $\rho_{\nu_1} = 4$; Hence, by Lemma 11,

$$4 = \zeta_{\nu_1} + \theta_{\nu_1}.$$

First assume that $\theta_{\nu_1} = 0$.

Then $\sigma = 2\nu_1$ and $f + B\nu_1 - 2\nu_1 = 0$.

$$4 = F(\nu_1) = (\nu_1 - 3)x_1 + 2(\nu_1 - 4)x_2 + 3(\nu_1 - 5)x_3 + \cdots .$$

To find the maximal ν_1, we suppose that $4 \geq x_1(\nu_1 - 3)$. Then $\nu_1 \geq 7$.

• case $\nu_1 = 7$. From hypothesis, it follows that $x_1 = 1$ and $x_j = 0$ if $j \neq 1$. Therefore, since $\sigma = 14, g_0 = 13^2 = 169$, we obtain

$$1 = x_1 = t_3 + t_6, t_3 + t_6 + t_2 + t_7 = 11, 3t_3 + 15t_6 + t_2 + 21t_7 = 169$$

$$2t_3 + 14t_6 + 20t_7 = 158; 6t_6 + 10t_7 = 78.$$

$3t_6 + 5t_7 = 39$; hence, $t_6 = 3$; a contradiction.

• case $\nu_1 = 6$. From hypothesis, it follows that $4 = F(6) = 3x_1 + 4x_2$. Then $t_4 = x_2 = 1$. Since $\sigma = 12, g_0 = 11^2 = 121$,

$$t_4 + t_2 + t_6 = 11, 6t_4 + t_2 + 15t_6 = 121,$$

$14t_6 = 105$, a contradiction.

• case $\nu_1 = 5$. From hypothesis, it follows that $4 = F(5) = 2x_1$. Then $t_3 + t_4 = x_1 = 2$. Since $\sigma = 10, g_0 = 9^2 = 81$,

$$t_3 + t_4 + t_2 + t_5 = 11, 3t_3 + 6t_4 + t_2 + 10t_5 = 81,$$

$2t_3 + 5t_4 + 9t_5 = 70, 3t_4 + 9t_5 = 66; t_4 + 3t_5 = 22$. Therefore, $t_4 = 1, t_3 = 1, t_5 = 7, t_2 = 2$ and the type is $[10 * 10; 5^7, 4, 3, 2^2]$ or its associates.

• case $\nu_1 = 4$. From hypothesis, it follows that $4 = F(4) = x_1$. Then $t_3 = x_1 = 4$. Since $\sigma = 8, g_0 = 7^2 = 49$, we obtain

$$t_3 + t_2 + t_4 = 11, 3t_3 + t_2 + 6t_4 = 49,$$

$2t_3 + 5t_4 = 38, 5t_4 = 30; t_4 = 6, t_3 = 4, t_2 = 1$. The type is $[8 * 8; 4^6, 3^4, 2]$ or its associates.

19.3.1 case $\theta_{\nu_1} > 0$

Second, assume that $\theta_{\nu_1} > 0, p > 0$.

By $4 = \rho_{\nu_1} > 3\nu_1 - 5$, $9 > 3\nu_1$, which contradicts the hypothesis $\nu_1 \geq 4$.

Third, assume that $\theta_{\nu_1} > 0, p = 0$. Then $\tilde{A} = 0$ and $4 = \zeta + \theta_{\nu_1}, \theta_{\nu_1} = \gamma = 2(\nu_1 - 2)(f + \nu_1 B - 2\nu_1) > 0$.

We shall study in the following cases 1) $B = 0$, 2) $B = 1$ and 3) $B \geq 2$, separately.

case 1) $B = 0$: $f = \sigma + u = 2\nu_1 + u, \gamma = 2(\nu_1 - 2)(f - 2\nu_1) = 2(\nu_1 - 2)u$.

case 2) $B = 1$: $f = \nu_1 + u, \gamma = 2(\nu_1 - 2)(f - \nu_1) = 2(\nu_1 - 2)u$.

case 3) $B = 2$: $f = u, \gamma = 2(\nu_1 - 2)f = 2(\nu_1 - 2)u$.

In any case,

$$4 = \zeta + \theta_{\nu_1} \geq 2(\nu_1 - 2)u \geq 4u \geq 4.$$

Thus $u = 1, \zeta = 0, \nu_1 = 4, \sigma = 8, g_0 = 56$.

$$t_3 = 0, t_2 + t_4 = 11, t_2 + 6t_4 = 56.$$

$5t_4 = 45; t_4 = 9, t_2 = 2$ and the type is $[8 * 9; 4^9, 2^2]$ or its associates.

19.4 case $\beta = 9$

Then $r = 12, \rho_{\nu_1} = 5 - \nu_1$; Hence, by Lemma 11,

$$5 - \nu_1 = \zeta_{\nu_1} + \theta_{\nu_1}.$$

First assume that $\zeta_{\nu_1} > 0$.

From $5 - \nu_1 \geq \zeta_{\nu_1} \geq \nu_1 - 3$, it follows that $\nu_1 = 4, \sigma = 8, g_0 = 49, 1 = \zeta_{\nu_1} = F(4) = x_1$.

$$t_3 = 1, t_3 + t_2 + t_4 = 12, 3t_3 + t_2 + 6t_4 = 49.$$

Hence, $2t_3 + 5t_4 = 37, 5t_4 = 35; t_4 = 7, t_3 = 1, t_2 = 4$. The type is $[8 * 8; 4^7, 3, 2^4]$ or its associates.

Second assume that $\zeta_{\nu_1} = 0$.

Suppose that $\tilde{A} > 0$.

Since $\tilde{A} + \gamma \geq 3\nu_1 - 5$, it follows that $5 - \nu_1 \geq 3\nu_1 - 5$; hence, $10 \geq 4\nu_1$, which contradicts the hypothesis $\nu_1 \geq 4$.

Suppose that $\tilde{A} = 0$. Then $5 - \nu_1 \geq 2(\nu_1 - 2)$; hence, $9 \geq 3\nu_1$, which contradicts the hypothesis $\nu_1 \geq 4$.

Therefore, we have established the classification of pairs (S, D) with $g(D) = 0, \kappa[D] = 2, P_2[D] = 3$ as follows :

Theorem 12 *Pairs (S, D) with $g(D) = 0, \kappa[D] = 2, P_2[D] = 3$ are classified as follows :*

1. *If $D^2 = -6$ then $r = 9$ and*
   (a) *if $\sigma = 15$ then the type is $[15 * 22, 1; 7^9]$.*
   (b) *If $\sigma = 16$ then the type is $[16 * 16; 8^6, 7^2, 6]$ or its associates.*
   (c) *If $\sigma = 20$ then the type is $[20 * 20; 10^7, 9, 5]$ or its associates.*
2. *If $D^2 = -7$ then $r = 10$ and*
   (a) *If $\sigma = 9$ then the type is $[9 * 13, 1; 4^{10}]$.*
   (b) *If $\sigma = 10$ then the type is $[10 * 10; 5^6, 4^3, 3]$ or its associates.*
   (c) *If $\sigma = 12$ then the type is $[12 * 12; 6^7, 5, 3^2]$ or $[12 * 12; 6^6, 5^3, 2]$ or their associates.*
   (d) *If $\sigma = 14$ then the type is $[14 * 14; 7^7, 6, 4, 2]$.*
3. *If $D^2 = -8$ then $r = 11$ and*
   (a) *if $\sigma = 7$ then the type is $[7 * 10, 1; 3^{11}]$.*
   (b) *If $\sigma = 8$ then the type is $[8 * 8; 4^6, 3^4, 2]$ or $[8 * 9; 4^9, 2^2]$ or their associates.*
   (c) *If $\sigma = 10$ then the type is $[10 * 10; 5^7, 4, 3, 2^2]$ or its associates.*
4. *If $D^2 = -9$ then $r = 12$ and the type is $[8 * 8; 4^7, 3, 2^4]$ or $[6 * 7; 3^9, 2^3]$ or their associates.*
5. *If $D^2 = -10$ then $r = 13$ and the type is $[6 * 6; 3^6, 2^7]$ or its associates.*
6. *If $D^2 = -11$ then $r = 14$ and the type is $[5 * 7, 1; 2^{14}]$.*
7. *If $D^2 = -12$ then $r = 15$ and the type is $[4 * 6; 2^{15}]$ or its associates.*

Note that pairs (S, D) with $g(D) > 0, P_2[D] = 3$ are enumerated as follows :

Proposition 21 *Pairs (S, D) with $g(D) > 0, \kappa[D] = 2, P_2[D] = 3$ are classified as follows :*

1. *If $\kappa[D] = 1$ then $g = 2$ the type is $[2 * 3; 1]$ or its associates, where $D^2 = Z^2 = 0$.*

2. *If $\kappa[D] = 2$ then $g = 1$ and*

   (a) *if $D^2 = -3$, then the type is $[12 * 12; 6^6, 5^3]$ or its associates.*
   (b) *If $D^2 = -4$, then the type is $[8 * 8; 4^6, 3^4]$ or $[8 * 9; 4^9, 2]$ or $[10 * 10; 5^7, 4, 3, 2]$ or their associates.*
   (c) *If $D^2 = -5$, then the type is $[6 * 7; 3^9, 2^2]$ or $[8 * 8; 4^7, 3, 2^3]$ or their associates.*
   (d) *If $D^2 = -6$, then the type is $[6 * 6; 3^6, 2^6]$ or its associates.*
   (e) *If $D^2 = -7$, then the type is $[5 * 7, 1; 2^{13}]$.*
   (f) *If $D^2 = -8$, then the type is $[4 * 6; 2^{14}]$ or its associates*

20 invariant ψ

The invariant ψ defined to be $\Omega - \omega$ is non-negative, if $\sigma \geq 6$ except for the type $[6 * 8, 1; 2^r]$. Next,we shall compute A, α, Ω and ω for pairs with $\nu_1 \leq 3$ as follows.

20.1 examples

If the type is $[\sigma * e, B; 3^{t_3}, 2^{t_2}]$, then letting f be $e - B\sigma$, we obtain

$$\begin{aligned}
D^2 &= \sigma\tilde{B} - 9t_3 - 4t_2 = \tau_0 - 9t_3 - 4t_2, \\
Z^2 &= (\sigma - 2)(\tilde{B} - 4) - 4t_3 - t_2 = \tau_2 - 4t_3 - t_2, \\
g &= \frac{(\sigma - 1)(\tilde{B} - 2)}{2} - 3t_3 - t_2 = \frac{\tau_1}{2} - 3t_3 - t_2.
\end{aligned}$$

Hence,

$$\begin{aligned}
A &= \tau_2 - \frac{\tau_0}{2} + 1 - t_3 && = \frac{\tau_3}{2} - 1 - t_3, \\
\alpha &= \sigma\tilde{B} - 4\sigma - 2\tilde{B} - 3t_3 && = \tau_2 - 8 - 3t_3, \\
\Omega &= \sigma\tilde{B} - 8\sigma - 4\tilde{B} + 24 + t_2 && = \tau_4 - 8 + t_2, \\
\omega &= \frac{\sigma\tilde{B} - 6\sigma - 3\tilde{B}}{2} + t_2 && = \frac{\tau_3}{2} - 9 + t_2.
\end{aligned}$$

Therefore,

$$\psi = \Omega - \omega = \sigma\tilde{B}/2 - 5\sigma - 5\tilde{B}/2 + 24 = (\sigma - 5)(\tilde{B} - 10)/2 - 1 = \tau_5/2 - 1.$$

If $\sigma \geq 6$ and $\psi < 0$ then $\tilde{B} = 10$ and therefore, the type is $[6 * 8, 1; 2^r]$ and in this case $\psi = -1$.

If $\sigma \geq 6$ and $\psi = 0$ then $\tau_5 = 2$; hence, either 1) $\sigma = 6, \tilde{B} = 12$ or 2) $\sigma = 7, \tilde{B} = 11$.

In the case 1), the type is $[6 * 6; 3^{t_3}, 2^{t_2}]$ or their associates;

In the case 2), the type is $[7 * 9, 1; 2^{t_2}]$ or their associates because $2 = f \geq \nu_1$.

If $\sigma \geq 6$ and $\psi = 1$ then $\tau_5 = 4$; hence, $\sigma = 6, \tilde{B} = 14$; the type is $[6 * 7; 3^{t_3}, 2^{t_2}]$ or their associates;

If $\sigma \geq 6$ and $\psi = 2$ then $\tau_5 = 6$; hence, 1) $\sigma = 6, \tilde{B} = 16$ or 2) $\sigma = 7, \tilde{B} = 13$ or 3) $\sigma = 8, \tilde{B} = 12$.

In case 1), the type is $[6 * 8; 3^{t_3}, 2^{t_2}]$ or their associates.

In case 2), the type is $[7 * 10, 1; 3^{t_3}, 2^{t_2}]$ or their associates;

In case 3), the type is $[8 * 10, 1; 2^r]$ or their associates.

In that follows , we assume that $\nu_1 \geq 4$.

20.2 pairs with small ψ

Under the assumption that $\sigma \geq 6$ and $\psi \geq 0$, we shall determine the type of pairs with small ψ. Say $\psi = 0, 1, 2$.

Putting $\overline{g} = g - 1$, we obtain

$$\omega = 3\overline{g} - D^2 \geq 0,\ \Omega = 3Z^2 - 4\overline{g} = \omega + \psi.$$

Thus,

$$D^2 = 3\overline{g} - \omega,\ Z^2 = \frac{4\overline{g} + \omega + \psi}{3}.$$

Since $\overline{g} + \omega + \psi = 3Z^2 - 3\overline{g} = 3A$,introduce a parameter k by $k = A - 1$; hence, $\overline{g} + \omega + \psi = 3k + 3$.

Then

$$D^2 = 4\overline{g} + \psi - 3k - 3,\ Z^2 = \overline{g} + k + 1,$$

and

$$8 - r = K_S^2 = Z^2 + D^2 - 4\overline{g} = \overline{g} + \psi - 2k - 2.$$

Hence,

$$\overline{g} - r = 2\overline{g} + \psi - 2k - 10.$$

Suppose that $k < 0$. Then $k = -1$ and 1) $g = 1, \omega = \psi = 0$ or 2) $g = 0, \omega + \psi = 1$. In the case 1), $\Omega = 0$. However, $\Omega = 3Z^2 - 4\overline{g} = 3Z^2 \geq 3$; a contradiction.

In the case 2), $\Omega = 1$. However, $1 = \Omega = 3Z^2 - 4\overline{g} = 3Z^2 + 4 \geq 4 \geq 4$; a contradiction.

Therefore, $k \geq 0$ and

$$\xi_0 = 8 - \frac{D^2}{2} + \overline{g} - r = \frac{\psi - k - 1}{2},$$

and thus

$$\xi_1 = 4\overline{g} - D^2 = 3k + 3 - \psi.$$

Then by Lemma, we obtain

$$\zeta_{\nu_1} = (\psi - k - 1)\nu_1 + 3k + 3 - \psi + \tilde{\eta}.$$

Supposing that $\nu_1 \geq 4$, we shall enumerate types of pairs satisfying the above equation under the hypothesis $\psi = 0, 1, 2$.

First, we note that if $\psi = 2$ then $k > 0$ or $g = 0$.

Claim 9 *If $\psi = 2$ then $k > 0$ or $g = 0$.*

Actually, suppose that $k = 0$. Then $\omega + \overline{g} = 1$. Hence, 1) $\omega = 1, \overline{g} = 0, \Omega = 3$ or 2) $\omega = 0, \overline{g} = 2, \Omega = 3$ or 3) $\omega = 2, \overline{g} = -1, \Omega = 4$.

In the case 1), $D^2 = -1, Z^2 = 1, r = 10 - 2 - \overline{g} = 8$. Therefore, $K_S^2 = 0$ and then by Riemann-Roch, $|-K_S| \neq \emptyset$. Hence, $-K_S \cdot (2Z - D) \geq 0$. But

$$0 \leq -K_S \cdot (2Z - D) = -(Z - D) \cdot (2Z - D) = -2Z^2 - D^2 = -1.$$

This is a contradiction.

In the case 2), $\Omega = 2, Z^2 = 2, g = 2$. But by the previous result, $Z^2 = g = 2$ implies that $\nu_1 = 2$, which contradicts the hypothesis $\nu_1 \geq 4$.

In the case 3), $D^2 = -5, Z^2 = 0, r = 9$. Therefore, since $Z^2 = 0$, it follows that $g = 0$.

Note that in the case 3), the type becomes either $[10 * 11; 5^9]$ or $[12 * 12; 6^7, 5, 4]$ or their associates.

20.3 case $p \geq 1$

First assume that $p \geq 1$. Then by Lemma, $\tilde{\eta} \leq (\delta_{1,B} + 2 - 2\nu_1)p$. Hence ,

$$\begin{aligned} 0 \leq \zeta_{\nu_1} &\leq (\psi - k - 1)\nu_1 + 3k + 3 - \psi + (\delta_{1,B} + 2 - 2\nu_1)p \\ &= (\psi - k - 3)\nu_1 + 3k + 5 + \delta_{1,B} - \psi + (p-1)(\delta_{1,B} + 2 - 2\nu_1) \\ &= (\psi - k - 3)(\nu_1 - 3) + (p-1)(\delta_{1,B} + 2 - 2\nu_1) - 4 + 2\psi + \delta_{1,B} \\ &\leq -4 + 2\psi + \delta_{1,B}. \end{aligned}$$

Therefore, since $\nu_1 \geq 4$, it follows that $\psi = 2, \nu_1 = 4, p = 1, \sigma = 9, k = 0, g = 0$. Hence, $3 = 3k + 3 = -1 + \omega + \psi = -1 + \omega + 2$. This implies that $\omega = 2, \omega = -3 - D^2$. Therefore, $D^2 = -5, g = 0$. But by Theorem , $\sigma = 10, 12$, which contradicts $\sigma = 9$. But $\nu_1 \geq 4$ is assumed.

20.4 case $p = 0$

Then $\eta = 2(\nu_1 - 2)(2\nu_1 - B\nu_1 - f) \leq 0$ and

$$0 \leq \zeta_{\nu_1} = (\psi - k - 1)\nu_1 + 3k + 3 - \psi + \eta.$$

If $\psi = 0$ then $0 \leq \zeta_{\nu_1} = (3 - \nu_1)(k+1) + \eta \leq (3 - \nu_1)(k+1)$. This implies that $\nu_1 = 3$ and $\eta = 0$. Hence, the type becomes $[6 * 6; 3^{t_3}, 2^{t_2}]$ or its associates. Note that $k = t_2 = \omega$.

20.5 case $\psi = 1$

If $\psi = 1$ then $0 \leq \zeta_{\nu_1} = -\nu_1 k + 3k + 2 + \eta$.

If $\eta \neq 0$ then $\eta \leq 4 - 2\nu_1$ and hence,

$$0 \leq -\nu_1 k + 3k + 2 + 4 - 2\nu_1 = -(k+2)\nu_1 + 3k + 6 = -(k+2)(\nu_1 - 3).$$

However, since $\nu_1 \geq 4$, it follows that $-(k+2) < 0$, a contradiction.

Suppose that $\eta = 0$,i.e., $\tilde{B} = 4\nu_1$ and $g_0 = (2\nu_1 - 1)^2$. Since $\zeta_{\nu_1} = -\nu_1 k + 3k + 2 = F(\nu_1)$, it follows that case A): $-\nu_1 k + 3k + 2 = F(\nu_1) = 0$ or case B): $-\nu_1 k + 3k + 2 = F(\nu_1) \geq \nu_1 - 3$.

In the case A),$-\nu_1 k + 3k + 2 = k(3 - \nu_1) + 2 = 0$. Hence, 1) $\nu_1 = 5, k = 1$ or 2)$\nu_1 = 4, k = 2$.

In the case B), it follows that $\nu_1 \leq \frac{3k+5}{k+1}$. In particular, if $k=0$ then $\nu_1 \leq 5$. Moreover, if $k=1$ then $\nu_1 \leq 4$.

- Suppose that $\nu_1 = 5$. In both cases A) and B), $k = 0,1,2$. Hence, $g_0 = 81, r = 10 - \psi - \overline{g} + 2k = 10 - g + 2k$

In the case A), $k = 1, r = 12 - g, t_3 = t_4 = 0$. By genus formula, we get

$$t_2 + t_3 + t_4 + t_5 = r = 12 - g, t_2 + 3t_3 + 6t_4 + 10t_5 = 81 - g.$$

So,

$$t_2 + t_5 = r = 12 - g, t_2 + 10t_5 = 81 - g; 9t_5 = 69.$$

This is a contradiction.

In the case B), $k = 0, r = 10 - g$ and $\zeta_{\nu_1} = 2 = (5-3)x_1$, which induces that $x_1 = 1, x_1 = t_3 + t_4$. By genus formula, we get

$$t_2 + t_3 + t_4 + t_5 = r = 10 - g, t_2 + 3t_3 + 6t_4 + 10t_5 = 81 - g.$$

Hence, $2t_3 + 5t_4 + 9t_5 = 71; 3t_4 + 9t_5 = 71 - 2 = 69$. Therefore, $t_4 + 3t_5 = 23$. Hence, $t_5 = 7, t_4 = 2, t_3 = -1$,that is a contradiction.

- If $\nu_1 = 4$, then $k = 0,1,2; g_0 = 49, r = 10 + 2k - 1 - \overline{g} = 10 + 2k - g$. Hence, $\zeta_{\nu_1} = 2 - k = x_1$, which induces that $x_1 = t_3 = 2 - k$. By genus formula, we get

$$t_2 + t_3 + t_4 = r = 10 + 2k - g, t_2 + 3t_3 + 6t_4 = 49 - g.$$

Hence, $2t_3 + 5t_4 = 39 - 2k; t_4 = 7, t_3 = 2 - k, t_2 = 1 + 3k - g$. The type becomes $[8*8; 4^7, 3^{2-k}, 2^{1+3k-g}]$ or its associates. Here,$\omega = 2 + t_2 = 3 + 3k - g, \Omega = 3 + t_2 = 4 + 3k - g$ and $\psi = \Omega - \omega = 1$. Moreover,$k = 0,1,2$ and $g \leq 7$.

20.6 case $\psi = 2$

First note that when $\psi = 2$, $k = 0$ implies that $g = 0$ and the type has already been eumerated. So we assume $k = 0$.

Second, since $\psi = 2$ and $p = 0$, it follows that

$$0 \leq \zeta_{\nu_1} = (1-k)\nu_1 + 3k + 1 + \eta.$$

- If $\eta \neq 0$ then $\eta = 2(\nu_1 - 2)(2\nu_1 - B\nu_1 - f) \leq -2(\nu_1 - 2)$ and thus

$$0 \leq \zeta_{\nu_1} \leq (1-k)\nu_1 + 3k + 1 - 2(\nu_1 - 2).$$

Then $(k+1)\nu_1 \leq 3k+5$, which implies that $\nu_1 \leq 4$.
• If $\nu_1 = 4$ then $k=1$ and $\zeta_4 = 4+\eta \leq 0$. Moreover, $2\nu_1 - B\nu_1 - f = -1$, i.e. $4B+f=9$ and so $\tilde{B}=18, \zeta_4=0$. This implies that $t_3=0, g_0=56$. By $r = 10+2k-\psi-\overline{g} = 11-g$ and genus formula

$$t_2 + t_4 = 11 - g,\ t_2 + 6t_4 = 56 - g.$$

Hence, $t_4 = 9, t_2 = 2-g$. Thus the type becomes $[8*9; 4^9, 2^{2-g}]$.

20.7 case $\eta = 0$

In this case, $\tilde{B} = 4\nu_1$.

From $\zeta_{\nu_1} = (1-k)\nu_1 + 3k + 1 = F(\nu_1)$, we obtain two cases: case A): $(1-k)\nu_1 + 3k + 1 = F(\nu_1) = 0$ and case B): $(1-k)\nu_1 + 3k + 1 = F(\nu_1) \neq 0$.

In the case A),$(\nu_1 - 3)(k-1) = 4$. Hence, 1) $\nu_1 = 7, k = 2$ or 2)$\nu_1 = 5, k = 3$ or 3) $\nu_1 = 4, k = 5$.

• If $\nu_1 = 7, k = 2$ then $g_0 = 169, r = 13-g$. Since $F(\nu_1) = 0$, it follows that $t_3 = \cdots = t_6 = 0$. By genus formula, $t_2 + t_7 = 13 - g, t_2 + 21t_7 = 169 - g$. Hence $20t_7 = 156$; a contradiction.

• If $\nu_1 = 5, k = 3$ then $g_0 = 81, r = 15-g$. Since $F(\nu_1) = 0$, it follows that $t_3 = t_4 = 0$. By genus formula, $t_2 + t_5 = 15 - g, t_2 + 10t_7 = 81 - g$. Hence $9t_5 = 66$; a contradiction.

• If $\nu_1 = 4, k = 5$ then $g_0 = 49, r = 19 - g$. Since $F(\nu_1) = 0$, it follows that $t_3 = 0$. By genus formula, $t_4 = 6, t_2 = 13 - g$. Thus the type is $[8*8; 4^6, 2^{13-g}]$ or their associates.

In the case B),

$$0 \leq \zeta_{\nu_1} = (1-k)\nu_1 + 3k + 1 = F(\nu_1) \geq \nu_1 - 3,$$

and so $\nu_1 \leq \dfrac{3k+4}{k} = 3 + \dfrac{4}{k} \leq 7$.

• If $\nu_1 = 7$, then $k = 1, \sigma = 14, \tilde{B} = 28$ and

$$4 = F(\nu_1) = F(7) = (7-3)x_1 + \cdots.$$

Thus $x_1 = t_3 + t_6 = 1, x_2 = x_3 = 0$. Since $r = 10 + 2k - \psi - g + 1$, we obtain

$$t_2 + t_3 + t_6 + t_7 = 11 - g, \ t_2 + 3t_3 + 15t_6 + 21t_7 = 169 - g.$$

From this, it follows that $3t_6 + 5t_7 = 39, t_6 = 0, 1$;a contradiction.

- If $\nu_1 = 6$, then $\sigma = 12, g_0 = 121$ and $k \leq \dfrac{4}{\nu_1 - 3} = \dfrac{4}{3}$. Hence $k = 1, \ 4 = F(\nu_1) = F(6) = 3x_1 + 4x_2$; thus $x_2 = t_4 = 1, x_1 = t_3 = t_5 = 0$. Since $r = 10 + 2k - \psi - g + 1 = 11 - g$, we obtain

$$t_2 + t_4 + t_6 = 11 - g, \ t_2 + 6t_4 + 15t_6 = 1 = 121 - g.$$

Then $14t_6 = 105$; a contradiction.

- If $\nu_1 = 5$, then $\sigma = 10, g_0 = 81$ and $k \leq \dfrac{4}{\nu_1 - 3} = \dfrac{4}{2} = 2$. Hence $k = 1, 2$. Further,$6 - 2k = F(\nu_1) = F(5) = 2x_1$; thus $x_1 = t_3 + t_4 = 3 - k$. By genus formula,

$$t_2 + t_3 + t_4 + t_5 = 9 + 2k - g, \ t_2 + 3t_3 + 6t_4 + 10t_5 = 81 - g.$$

Hence, $3t_4 + 9t_5 = 66$. Then $t_5 = 7, t_4 = 1, t_3 = 2 - k, t_2 = 3k - g - 1$. The type becomes $[10*10; 5^7, 4, 3^{2-k}, 2^{3k-g-1}]$ or its associates. In this case, $\Omega = 7 - g, \omega = 3k - g - 1$.

- If $\nu_1 = 4$, then $\sigma = 8, g_0 = 49$ and $k \leq \dfrac{4}{\nu_1 - 3} = \dfrac{4}{1} = 4$. Hence $k = 1, 2, 3, 4$. Further,$5 - k = F(\nu_1) = F(4) = x_1$; thus $x_1 = t_3 = 5 - k$. By genus formula,

$$t_2 + t_3 + t_4 = r = 9 + 2k - g, \ t_2 + 3t_3 + 6t_4 = 49 - g.$$

Then $k = 1, t_4 = 6, t_3 = 5 - k = 4, t_2 = 3k - g - 2 = 1 - g$. Thus the type becomes $[8*8; 4^6, 3^{5-k}, 2^{3k-2-g}]$ or its associates, where $g \leq 1$. In this case, $\Omega = 6, \omega = 4$.

20.8 classification by $P_{3,1}[D]$

Consequently, we obtain the following result.

Theorem 13 *Suppose that a minimal pair (S, D) is derived from a $\#-$minimal pair (Σ_B, C) of type $[\sigma * e, B; \nu_1, \cdots, \nu_r]$ where $\sigma \geq 3$ or (S, D) is just $(\mathbf{P}^2, D)$ of type $[d; 1]$ where $d \geq 9$.*

1. *case* $P_{3,1}[D]=0$. *Then either* $\sigma \le 5$ *or the type is* $[6*8,1;2^r]$, *where* $g=20-r$.

2. *case* $P_{3,1}[D]=1$. *Then*

   (a) *the type becomes* $[6*6;3^{t_3},2^{t_2}]$, *where* $t_3 \le 8, t_2 = 25-3t_3-g$ *or their associates or*

   (b) $[7*9,1;2^{27-g}]$.

   (c) *The type is* $[9;1]$.

3. *case* $P_{3,1}[D]=2$. *Then*

   (a) *the type becomes* $[6*7;3^{t_3},2^{t_2}]$ *or their associates or*

   (b) $[8*8;4^7,3^{2-k},2^{1+3k-g}]$ *or their associates.*

4. *case* $P_{3,1}[D]=3$.

   (a) *If* $\sigma=6$ *then the type becomes* $[6*8;3^{t_3},2^{t_2}]$ *or its associates.*

   (b) *If* $\sigma=7$ *then the type becomes* $[7*10,1;3^{t_3},2^{t_2}]$.

   (c) *If* $\sigma=8$ *then the type becomes*

      i. $[8*8;4^6,3^{5-k},2^{3k-g-2}]$, *where* $k\le 5, g\le 3k-2$ *or*

      ii. $[8*9;4^9,2^{2-g}]$, *where* $g\le 2$ *or their associates or*

      iii. $[8*10,1;2^g]$, *where* $g\le 35$ *or their associates.*

   (d) *If* $\sigma=10$ *then the type becomes*

      i. $[10*10;5^7,4,3^{2-k},2^{3k-g-1}]$ *or its associates, where* $g\le 2$,*or*

      ii. $[10*11;5^9]$ *where* $g=0$ *or its associates.*

   (e) *If* $\sigma=12$ *then the type becomes* $[12*12;6^7,5,4]$ *or its associates where* $g=0$.

   (f) *The type is* $[10;1]$.

21 relations between Z^2 and D^2

Next, we shall study relations between Z^2 and D^2. First , we suppose that $\nu_1 \le 2$.

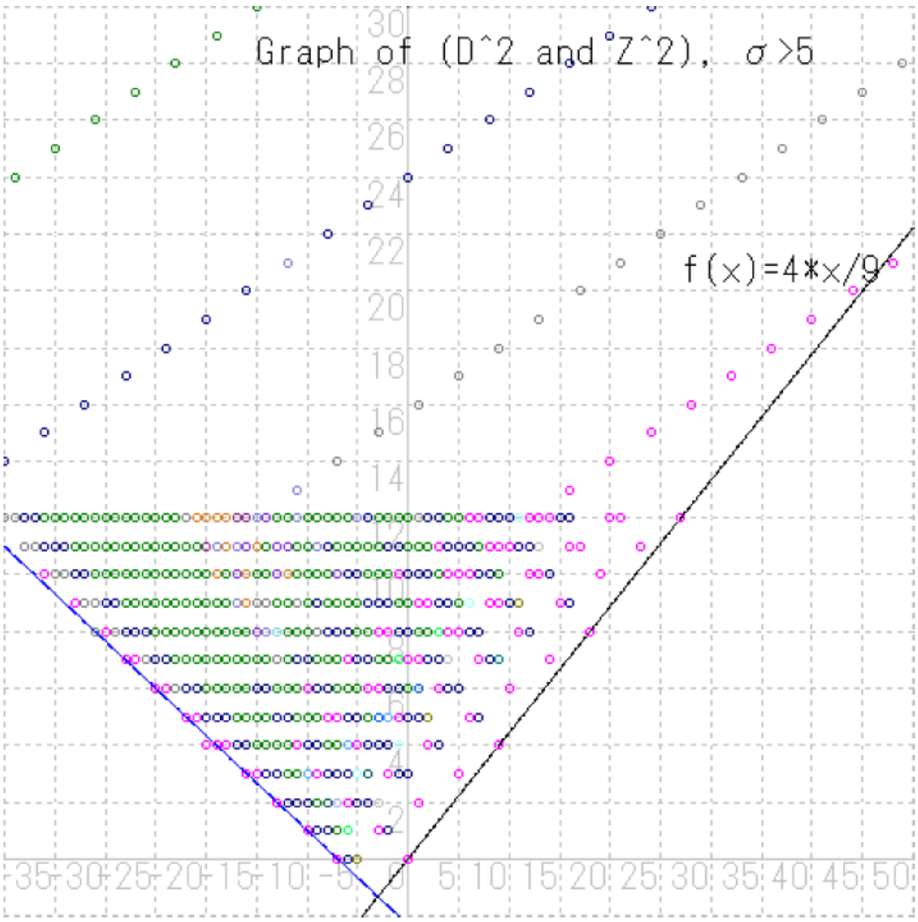

Figure 5: relations between D^2 and Z^2, with $\sigma \geq 6$

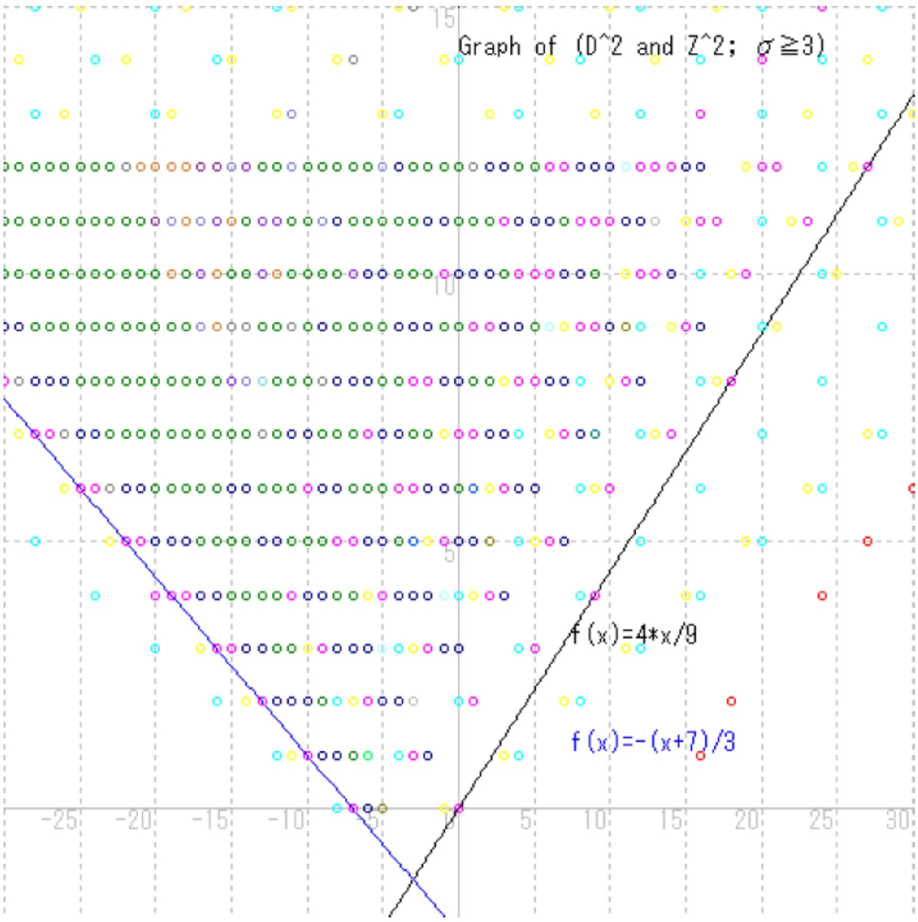

Figure 6: relations between D^2 and Z^2, with $\sigma \geq 6$

21.1 Case $\nu_1 \leq 2$

If a minimal pair (S, D) with $\kappa[D] = 2$ is derived from a $\#-$ minimal model of type $[\sigma * e, B; 2^r]$, then

$$D^2 = \sigma\tilde{B} - 4r,\ 2\overline{g} = \sigma\tilde{B} - 2\sigma - \tilde{B} - 2r,$$
$$Z^2 = K_S^2 - D^2 + 4\overline{g} = 8 - r - \sigma\tilde{B} + 4r + 2(\sigma\tilde{B} - 2\sigma - \tilde{B} - 2r),$$
$$Z^2 = 8 - r + (\sigma - 2)\tilde{B} - 4\sigma.$$

Eliminating $\overline{g}$ and $\tilde{B}$ from these, we obtain

$$\sigma Z^2 = (\sigma - 2)D^2 + (3\sigma - 8)r - 4\sigma(\sigma - 2).$$

In particular, if $\sigma = 3$ then

$$3Z^2 = D^2 - 12.$$

If $\sigma = 4$ then

$$2Z^2 = D^2 + 2r - 16.$$

If $\sigma = 5$ then

$$5Z^2 = 3D^2 + 7r - 40.$$

21.2 Case $\sigma \geq 6$

Hereafter, we suppose that $\sigma \geq 6$. If the type is not $[6 * 8, 1; 2^r]$ then by Theorem $|3Z - 2D| \neq \emptyset$ and so

$$2\psi = 2(3Z^2 - 7\overline{g} + D^2) = (3Z - 2D) \cdot (2Z - D) \geq 0.$$

Therefore,

$$3Z^2 + D^2 \geq 7\overline{g} \geq -7.$$

Furthermore, define $\Xi = 9Z^2 - 4D^2$. Then from

$$3\psi = 3(3Z^2 + D^2 - 7\overline{g}) = 9Z^2 + 3D^2 - 21\overline{g} = \Xi - 7(3\overline{g} - D^2)$$

it follows that

$$\Xi = 3\psi + 7\omega$$

which is nonnegative when $\sigma \geq 7$ asn the type is not $[6 * 8, 1; 2^r]$. Hence, in this case,

$$9Z^2 \geq 4D^2.$$

Moreover,if $9Z^2 - 4D^2 = 0$ then $\omega = 0, \psi = 0$. Then $3Z - 2D \sim D + 3K \sim 0$.

Furthermore,if$\Xi = 9Z^2 - 4D^2 > 0$ then $\Xi = 3, 6, 7 \cdots$.

21.3 Plane curves with only double points

Suppose that the type is $[d; 2^r]$. Then

$$Z^2 = (d-3)^2 - r, D^2 = d^2 - 4r.$$

Hence,

$$3Z^2 + D^2 + 7 = \frac{d^2 - 15d + 54}{2} + 7g, \quad 9Z^2 - 4D^2 = (d-9)(5d-9) + 7r.$$

If $d = 8$ then

$$3Z^2 + D^2 + 7 = -1 + 7g, \quad 9Z^2 - 4D^2 = 7r - 31.$$

Therefore, if the type is $[6{*}8, 1; 2^r]$ with $r < 5$, then $9Z^2 - 4D^2 = 7r - 31 < 0$.

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