ON THE NONBIRATIONAL INVARIANT e OF ALGEBRAIC PLANE CURVES

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Contents

1. CASE IN WHICH $B \geq 3$

Lemma 1.

(1) *If* $B \geq 2$ *then* $(2D + \sigma K_S) \cdot D \geq (2e - \sigma B - 2\sigma)\sigma \geq \sigma^2(B-2)$.
(2) *If* $B \geq 3$ *then* $(2D + \sigma K_S) \cdot D \geq \sigma^2$*; in particular,* $2\sigma\overline{g} - (\sigma - 2)D^2 \geq \sigma^2$*. Hence,* $\sigma\omega_1 + 2D^2 \geq \sigma^2$.

By Lemma 1(1), we get

$$(\sigma - 2)\omega_1 + 4\overline{g} = (2D + \sigma K_S) \cdot D \geq (2e - \sigma B - 2\sigma)\sigma,$$

where $\omega_1 = 2\overline{g} - D^2$.

By the way, if $B \geq 2$ then $e = \sigma B + u$; thus $\sigma = \frac{e-u}{B}$ and

$$2e - \sigma B - 2\sigma = (1 - \frac{2}{B})e + (1 + \frac{2}{B})u.$$

Therefore,

$$(\sigma - 2)\omega_1 + 4\overline{g} \geq ((1 - \frac{2}{B})e + (1 + \frac{2}{B})u)\sigma \geq (1 - \frac{2}{B})e\sigma. \qquad (1)$$

Hence,

$$(1 - \frac{2}{B})e \leq \frac{\sigma - 2}{\sigma}\omega_1 + \frac{4\overline{g}}{\sigma}.$$

Thus,

$$(1 - \frac{2}{B})e \leq \frac{\sigma - 2}{\sigma}\omega - (1 - \frac{6}{\sigma})\overline{g}.$$

Hence, if $B \geq 3, g > 0$ and $\sigma \geq 6$, then

$$(1 - \frac{2}{B})e \leq \frac{\sigma - 2}{\sigma}\omega,$$

and so

$$e \leq \frac{B}{B-2}\frac{\sigma - 2}{\sigma}\omega < \frac{B}{B-2}\omega \leq 3\omega. \qquad (2)$$

Thus we obtain the next result:

Proposition 1. *If* $g > 0, \sigma \geq 7$ *and* $B \geq 3$ *then* $(1 - \frac{2}{B})e < \omega$.

1.1. case when $g = 0$.

When $g = 0$, we get

$$(\sigma - 2)(\omega + 1) - 4 = (\sigma - 2)\omega_1 + 4\overline{g} \geq ((1 - \frac{2}{B})e + (1 + \frac{2}{B})u)\sigma.$$

From this, it follows that

$$\frac{(\sigma - 2)(\omega + 1) - 4}{\sigma} = (1 - \frac{2}{\sigma})\omega + 1 - \frac{6}{\sigma} \geq (1 - \frac{2}{B})e + (1 + \frac{2}{B})u.$$

We shall prove the following result.

Proposition 2. *If* $g = 0, \sigma \geq 7$ *and* $B \geq 3$ *then* $(1 - \frac{2}{B})e < \omega$.

Supposing that $(1-\frac{2}{B})e \geq \omega$, we shall derive $u=0$. Actually, by hypothesis,

$$\frac{(\sigma-2)(\omega+1)-4}{\sigma} = (1-\frac{2}{\sigma})\omega + 1 - \frac{6}{\sigma} \geq \omega + (1+\frac{2}{B})u.$$

Thus

$$1 > \frac{2\omega}{\sigma} + \frac{6}{\sigma} + u + \frac{2}{B} > u \geq 0. \quad (3)$$

Therefore, $u=0$; thus $e=\sigma B$ and so $\sigma=\frac{e}{B}$. Moreover, recalling the inequality (1),

$$(\frac{e}{B}-2)\omega_1 - 4 = (\sigma-2)\omega_1 + 4\overline{g} \geq (1-\frac{2}{B})\frac{e^2}{B},$$

and so

$$(e-2B)\omega > (1-\frac{2}{B})e^2 + 6B - e.$$

Finally, we obtain

$$\omega > (1-\frac{2}{B})e + 2B - 5 + \frac{4B(B-1)}{e-2B} > (1-\frac{2}{B})e. \quad (4)$$

Therefore, if $B \geq 3$,then

$$e < \frac{B}{B-2}\omega \leq 3\omega.$$

q.e.d.

In that follows, we assume $B \leq 2$.

2. CASE IN WHICH $\lambda \geq 1$

First we assume $\lambda = k - \omega_1 \geq 1$ and $k > 0$.

2.1. case in which $\widetilde{\mathcal{Z}} > 0$.

Suppose that $\widetilde{\mathcal{Z}} = \nu_1 Y - X > 0$. Then

$$\widetilde{\mathcal{Z}} \geq \nu_1 - 1.$$

Recalling that

$$\widetilde{\mathcal{Z}} = \nu_1 Y - X = -\nu_1\lambda - \tilde{k} - \omega_1 + 2\overline{g},$$

we obtain

$$\nu_1 - 1 \leq -\nu_1\lambda - \tilde{k} - \omega_1 + 2\overline{g}.$$

Thus,

$$\nu_1 + \lambda\nu_1 - 1 \leq -\tilde{k} - \omega_1 + 2\overline{g},$$

and

$$2\nu_1 \leq \nu_1 - \lambda\nu_1 + 1 - \tilde{k} - \omega_1 + 2\overline{g}. \quad (5)$$

Hence,

$$\sigma = 2\nu_1 + p \leq (1-\lambda)\nu_1 + 1 + p - \tilde{k} - \omega_1 + 2\overline{g}.$$

We distinguish the various cases according to the value of B.

1) $B = 0$.
Then by hypothesis, recalling that $\nu_1 \geq 2$,

$$\begin{aligned} e = \sigma + u = 2\nu_1 + p + u \\ &\leq (1-\lambda)\nu_1 + 1 + p - \tilde{k} - \omega_1 + 2\overline{g} + u \\ &\leq 2(1-\lambda) + 1 + p - \tilde{k} - \omega_1 + 2\overline{g} + u \\ &= 2(1-k+\omega_1) + 1 + p - \tilde{k} - \omega_1 + 2\overline{g} + u. \\ &= 3 - 2k + p - \tilde{k} + \omega_1 + 2\overline{g} + u. \end{aligned}$$

Therefore, by $k = 4p + 2u$,

$$\begin{aligned} \omega_1^2 + \omega_1 + 2g - e &\geq \omega_1^2 + \omega_1 + 2g - (1 - 2k + p - \tilde{k} + \omega_1 + 2g + u) \\ &= \omega_1^2 + \tilde{k} + 2k - 1 - p - u \\ &= \omega_1^2 + k - 1 + \tilde{k} + 3p + u \\ &\geq 0. \end{aligned}$$

Thus we get

$$e \leq \omega_1^2 + \omega_1 + 2 + 2\overline{g}. \tag{6}$$

2) $B = 1$.
Then by hypothesis, recalling that $\nu_1 \geq 2$,

$$\begin{aligned} e = \sigma + u + \nu_1 = 3\nu_1 + p + u \\ &\leq \frac{3}{2}((1-\lambda)\nu_1 + 1 - \tilde{k} - \omega_1 + 2\overline{g}) + u + p \\ &\leq \frac{3}{2}(2(1-\lambda) + 1 - \tilde{k} - \omega_1 + 2\overline{g}) + u + p \\ &\leq \frac{3}{2}(3 - 2k + \omega_1 - \tilde{k} + 2\overline{g}) + u + p. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{3}{2}(\omega_1^2 + 2 + 2\overline{g}) - e \\ &\qquad \geq \frac{3}{2}(\omega_1^2 + \omega_1 + 2 + 2\overline{g}) - (\frac{3}{2}(3 - 2k + \omega_1 - \tilde{k} - \omega_1 + 2\overline{g}) + u + p) \\ &\qquad \geq \frac{3}{2}(\omega_1^2 - 1 + \tilde{k}) + 3k - p - u \\ &\qquad \geq 2k + 2p + u > 4. \end{aligned}$$

Thus we get

$$e < \frac{3}{2}(\omega_1^2 + \omega_1 + 2 + 2\overline{g}). \tag{7}$$

3) $B = 2$.
By the similar argument, we get

$$e < 2(\omega_1^2 + \omega_1 + 2 + 2\overline{g}). \tag{8}$$

3. CASE IN WHICH $\widetilde{\mathcal{Z}} = 0$

Suppose that $\nu_1 Y - X = \widetilde{\mathcal{Z}} = 0$. Then $\nu_1 = \cdots = \nu_r$ and hence,$X = r\nu_1^2, Y = r\nu_1$. Thus

- $(r-8)\nu_1^2 = 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $(r-8)\nu_1 = k + \omega_1$.

3.1. case in which $k > 0$.

Hereafter, we suppose that $k > 0$.

3.2. case in which $r \geq 9$.

(1) Suppose that $r \geq 9$. Then $\nu_1 \leq (r-8)\nu_1 = k + \omega_1$. Thus,

$$\nu_1 \leq k + \omega_1.$$

We distinguish the various cases according to the value of B.

(i) $B = 0$.
In this case,

$$e = \sigma + u = 2\nu_1 + p + u \leq 2k + 2\omega_1 + p + u.$$

Since

$$2\lambda \leq \nu_1\lambda \leq -\tilde{k} - \omega_1 + 2\overline{g}, \tag{9}$$

it follows that

$$\begin{aligned} \omega_1^2 + \omega_1 + 2\overline{g} + 2 &\geq \omega_1^2 + \omega_1 + \nu_1\lambda + \tilde{k} + \omega_1 + 2 \\ &\geq \omega_1^2 + \omega_1 + 2\lambda + \tilde{k} + \omega_1 + 2 \\ &= \omega_1^2 + \omega_1 + 2k - 2\omega_1 + \tilde{k} + \omega_1 + 2 \\ &= \omega_1^2 + 2 + \tilde{k} + 2k. \end{aligned}$$

Hence,

$$\omega_1^2 + \omega_1 + 2\overline{g} + 2 \geq \omega_1^2 + 2 + \tilde{k} + 2k. \tag{10}$$

Furthermore, from (9) and (10), it follows that

$$\begin{aligned}\omega_1^2+\omega_1+2\overline{g}+2-e \geq& \omega_1^2+2+\tilde{k}+2k-(2k+2\omega_1+p+u)\\ \geq& \omega_1^2-2\omega_1+2+\tilde{k}-p-u\\ =&(\omega_1-1)^2+1+\tilde{k}-p-u.\end{aligned}$$

Here, if $p>0$ then $\tilde{k}-p-u=p(4p+2u-2p)-p-u=2p^2-p+u(2p-1)>0$. However, if $\nu_1=2$ and $p=0$ then $\sigma=4$ and $e=4+u$. Moreover,

$$\begin{aligned}\omega_1^2+\omega_1+2\overline{g}+2-e \geq& \omega_1^2+2+2k-(4+u)\\ \geq& 2+4u-4-u\\ =&3u-2>0.\end{aligned}$$

Therefore,if $\nu_1\geq 2$, then

$$e<\omega_1^2+\omega_1+2+2\overline{g}.$$

(ii) $B=1$.
In this case,

$$e=\sigma+u+\nu_1=3\nu_1+p+u\leq 3k+3\omega_1+p+u.$$

Furthermore, by (10), we obtain

$$\begin{aligned}\frac{3}{2}(\omega_1^2+\omega_1+2\overline{g}+2)-e \geq& \frac{3}{2}(\omega_1^2+2+\tilde{k}+2k)-\frac{3}{2}(2k+2\omega_1)-p-u\\ \geq& \frac{3}{2}(\omega_1^2-2\omega_1+2+\tilde{k})-p-u.\end{aligned}$$

By the same argument as above, we conclude that the last term is positive and hence,

$$e<\frac{3}{2}(\omega_1^2+\omega_1+2+2\overline{g}).$$

(iii) $B=2$.
In this case, we obtain

$$e<2(\omega_1^2+\omega_1+2+2\overline{g}). \tag{11}$$

3.3. **case in which** $r = 8$.

(2)

Suppose that $r = 8$. Then

- $0 = (r-8)\nu_1^2 = 2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}$,
- $0 = (r-8)\nu_1 = k + \omega_1$.

Hence,$\omega_1 = -k \leq -3$. Furthermore, $\lambda = k - \omega_1 = 2k$ and

$$2\nu_1 = \frac{-\tilde{k} - \omega_1 + 2\overline{g}}{k}.$$

We distinguish the various cases according to B.

(i) $B = 0$.

Then,

$$e = \sigma + u = 2\nu_1 + p = \frac{-\tilde{k} - \omega_1 + 2\overline{g}}{k} + p + u.$$

Moreover,

$$\begin{aligned}
\omega_1^2 + \omega_1 + 2 + 2\overline{g} - e &= \omega_1^2 + \omega_1 + 2 + 2\overline{g} - (\frac{-\tilde{k} - \omega_1 + 2\overline{g}}{k} + p + u) \\
&= k(k-1) + 1 + 2\overline{g}(1 - \frac{1}{k}) + \frac{\tilde{k}}{k} - p - u \\
&\geq k(k-1) + 1 + (2\lambda + \tilde{k} + \omega_1)(1 - \frac{1}{k}) + \frac{\tilde{k}}{k} - p - u \\
&= k(k-1) + 1 + (4k + \tilde{k} - k)(1 - \frac{1}{k}) + \frac{\tilde{k}}{k} - p - u \\
&= k^2 + k - 2 + \tilde{k} + 3p + u > 0.
\end{aligned}$$

Thus

$$\omega_1^2 + \omega_1 + 2 + 2\overline{g} - e > 0.$$

(ii) $B = 1$.

Then,

$$e = \sigma + u + \nu_1 = 3\nu_1 + p + u = \frac{3}{2}(\frac{-\tilde{k} - \omega_1 + 2\overline{g}}{k}) + p + u.$$

Moreover,

$$\begin{aligned}
\frac{3}{2}(\omega_1^2 + \omega_1 + 2 + 2\overline{g}) - e &= \frac{3}{2}(\omega_1^2 + \omega_1 + 2 + 2\overline{g}) - \frac{3}{2}(\frac{-\tilde{k} - \omega_1 + 2\overline{g}}{k}) - p - u \\
&= k^2 + k - 2 + \tilde{k} + 3p + u > 0.
\end{aligned}$$

Thus
$$\frac{3}{2}(\omega_1^2 + \omega_1 + 2 + 2\overline{g}) - e > 0.$$

(iii) $B = 2$.
Then,
$$e = 2\sigma + u = 4\nu_1 + 2p = 4(\frac{-\tilde{k} - \omega_1 + 2\overline{g}}{k}) + 2p + u.$$
Moreover,
Thus
$$2(\omega_1^2 + \omega_1 + 2 + 2\overline{g}) - e > 0.$$

3.4. **case in which** $r \le 7$.
(3) Suppose that $r \le 7$. Then letting s be $8 - r > 0$, we get

- $s\nu_1^2 = -2k\nu_1 - \tilde{k} - \omega_1 + 2\overline{g}$,
- $s\nu_1 = -k - \omega_1$.

Since $\nu_1 \le s\nu_1 = -k - \omega_1$, it follows that
$$\sigma \le -2k - 2\omega_1 + p.$$
We distinguish the various cases according to B.

(i) $B = 0$.
Then,
$$e = \sigma + u \le -2k - 2\omega_1 + p + u.$$
Moreover,
$$\begin{aligned}\omega_1^2 + \omega_1 + 2 + 2\overline{g} - e &\ge \omega_1^2 + 3\omega_1 + 2 + 2\overline{g} + 2k - p - u \\ &\ge \omega_1(3 + \omega_1) + 2 + 2\overline{g} + (8 - 1)p + 3u.\end{aligned}$$

The function defined by
$$F_0(x) = x(3 + x) + 2 + 2\overline{g} + (2w - 1)p + 3u \tag{12}$$
has minimal values at $x = -1$ or -2. By
$$F_0(-1) = F_0(-2) = -2 + 2 + 2\overline{g} + (2w - 1)p + 3u \ge (2w - 1)p + 3u - 2$$
$F_0(x) > 0$ if $k > 0$.

(ii) $B = 1$.
Then,
$$e = \sigma + \nu_1 + u \le -3k - 3\omega_1 + p + u.$$
Moreover,

$$\begin{aligned}\frac{3}{2}(\omega_1^2+\omega_1+2+2\overline{g})-e &\geq \frac{3}{2}(\omega_1^2+3\omega_1+2+2\overline{g}+2k)-p-u\\ &\geq \frac{3}{2}(\omega_1(3+\omega_1))+3+3\overline{g}+8p+3u\\ &\geq -3+3\overline{g}+8p+3u\\ &\geq 2.\end{aligned}$$

3.5. **case in which $k=0$.**

(4) Suppose that $k=0$. Then

- $(r-8)\nu_1^2=\omega_1-2\overline{g}$,
- $(r-8)\nu_1=\omega_1$.

Recall that $\lambda=-\omega_1\geq 1$. Then $\overline{g}-\omega=-\omega_1\geq 1$ and so $\overline{g}>0, r<8$. Letting $s=8-r$, we get

- $s\nu_1^2=-\omega_1+2\overline{g}$,
- $s\nu_1=-\omega_1$.

Then $\sigma=2\nu_1=\frac{-2\omega_1}{s}$ and so

$$\begin{aligned}\omega_1^2+\omega_1+2+2\overline{g}-\sigma &= \omega_1^2+\omega_1+2+2\overline{g}+\frac{2\omega_1}{s}\\ &= (\omega_1+\frac{s+2}{2s})^2+2+2\overline{g}-(\frac{s+2}{2s})^2\end{aligned}$$

which is positive.

We distinguish the various cases according to B.

(i) $B=0$.

Then $e=\sigma=2\nu_1$ and

$$\begin{aligned}\omega_1^2-\omega_1+2\overline{g}-e &= s^2\nu_1^2-2s\nu_1+s\nu_1^2+2-e\\ &= s^2\nu_1^2-2s\nu_1+s\nu_1^2+2-(2\nu_1)\\ &= (s^2+s)\nu_1^2-2(s+1)\nu_1+2\end{aligned}$$

But since $\nu_1\geq 2$, it follows that

$$(s^2+s)\nu_1^2-2(s+1)\nu_1+2\geq 4(s^2+s)-4(s+1)+2=4s^2-2\geq 2.$$

(ii) $B=1$.

Then $e=\sigma+\nu_1=3\nu_1$ and

$$\frac{3}{2}(\omega_1^2-\omega_1+2\overline{g})-e=\frac{3}{2}(\omega_1^2-\omega_1+2\overline{g}-2\nu_1)\geq 3.$$

Therefore,

$$e \geq \frac{3}{2}(\omega_1^2 - \omega_1 + 2\overline{g}) - 3.$$

(iii) $B = 2$.
Then $e = 2\sigma = 4\nu_1$ and so

$$e \geq 2(\omega_1^2 - \omega_1 + 2\overline{g}) - 1.$$

4. CASE IN WHICH $\lambda \leq 0$

Suppose that $k > 0$. By hypothesis, $\lambda = k - \omega_1 \leq 0$.
By applying Lemma of Tanaka and Matsuda to this,

$$V = (k + \omega_1)^2 - (2k\nu_1 + \tilde{k} + \omega_1 - 2\overline{g}) \geq 0.$$

Thus

$$\sigma = 2\nu_1 + p \leq k + 2\omega_1 + \frac{\omega_1^2 - \omega_1 + 2\overline{g}}{k} + p. \tag{13}$$

We distinguish the various cases according to B.

(1) $B = 0$.
Then , $e = \sigma + u$ and

$$e = \sigma + u \leq k + 2\omega_1 + \frac{\omega_1^2 - \omega_1 + 2\overline{g}}{k} + p + u. \tag{14}$$

We shall prove the following

$$e \leq \omega_1^2 + \omega_1 + 2\overline{g} + 2.$$

By 5 and $\omega_1 \geq k$, we get

$$\begin{aligned}
\omega_1^2 + \omega_1 + 2\overline{g} + 2 - e &\geq (1 - \frac{1}{k})(\omega_1^2 - \omega_1 + 2\overline{g}) + k - p - u - 2k + 2 \\
&\geq (1 - \frac{1}{k})(k^2 - k + 2\overline{g}) + 3p + 2u - 2k + 2 \\
&= (k-2)^2 - 1 + 2(1 - \frac{1}{k})\overline{g} + 3p + 2u.
\end{aligned}$$

If $k = 2$ then $p = 0, u = 1$ and so

$$(k-2)^2 - 1 + 2(1 - \frac{1}{k}\overline{g} + 2 = 1 + \overline{g} = g \geq 0.$$

Otherwise $(k-2)^2 - 1 + 2(1 - \frac{1}{k})\overline{g}) + 3p + 2u > 0$, that completes the proof.

(2) $B = 1$.
Then , $e = \sigma + u + \nu_1$ and

$$e = \sigma + u + \nu_1 = 3\nu_1 + p + u \leq k + \frac{3}{2}(2\omega_1 + \frac{\omega_1^2 - \omega_1 + 2\overline{g}}{k}) + p + u. \tag{15}$$

We shall prove the following

$$e \leq \frac{3}{2}(\omega_1^2 + \omega_1 + 2\overline{g} + 2).$$

5. ESTIMATE OF e IN TERMS OF A

Replacing σ by $\frac{e-u}{B}$, we get

$$2e - B\sigma - 2\sigma = e(1 - \frac{2}{B}) + u(1 + \frac{2}{B})$$

and then

$$\begin{aligned} \sigma A_1 + 4\overline{g} &= \sigma A - (\sigma - 4)\overline{g} \\ &\geq (2e - B\sigma - 2\sigma)(\sigma - 2) \\ &= e(1 - \frac{2}{B})(\sigma - 2) + u(1 + \frac{2}{B})(\sigma - 2). \end{aligned}$$

Hence, supposing that $\sigma \geq 5$, we get

$$\sigma A - (\sigma - 4)\overline{g} \geq e(1 - \frac{2}{B})(\sigma - 2) + u(1 + \frac{2}{B})(\sigma - 2). \tag{16}$$

Thus, when $g > 0$,

$$\sigma A \geq e(1 - \frac{2}{B})(\sigma - 2); \tag{17}$$

in other words,

$$e \leq \frac{B\sigma}{(B-2)(\sigma-2)}. \tag{18}$$

5.1. case when $g = 0$.

Next, we shall show the inequality : $e \leq \frac{B\sigma}{(B-2)(\sigma-2)}$ even if $g = 0$.

First assume that $g = 0$ and $e > \frac{B\sigma}{(B-2)(\sigma-2)}A$.

From the inequality

$$\sigma A \geq e(1 - \frac{2}{B})(\sigma - 2) + (1 + \frac{2}{B})u(\sigma - 2) + \sigma - 4\overline{g}, \tag{19}$$

it follows that

$$e\frac{(B-2)(\sigma-2)}{B} > \sigma A \geq e(1 - \frac{2}{B})(\sigma - 2) + (1 + \frac{2}{B})u(\sigma - 2) + \sigma - 4\overline{g}.$$

However, since $g = 0$,it follows that

$$0 \geq (1 + \frac{2}{B})u(\sigma - 2) + (\sigma - 4\overline{g})(\sigma - 2) = (1 + \frac{2}{B})u(\sigma - 2) + 4 - \sigma.$$

Hence,

$$1 > \frac{\sigma - 4}{(\sigma - 2)} > u + \frac{2u}{B} \geq u \geq 0. \tag{20}$$

Therefore, $u = 0$ and so $e = B\sigma$. Hence,

$$\sigma A \geq e(1 - \frac{2}{B})(\sigma - 2) + 4 - \sigma = (B-2)\sigma(\sigma - 2) + 4 - \sigma.$$

In other words,

$$A \geq (B-2)(\sigma-2) + \frac{4}{\sigma} - 1.$$

By hypothesis, $-1 < \frac{4}{\sigma} - 1 \leq 0$. Hence, by $e = B\sigma$, we get

$$A \geq (B-2)(\sigma-2) = e\frac{(\sigma-2)(B-2)}{B\sigma}.$$

Consequently,

$$e \leq \frac{B\sigma}{(B-2)(\sigma-2)}A. \tag{21}$$

If $B \geq 4$ and $\sigma \geq 6$, then

$$e \leq \frac{B\sigma}{(B-2)(\sigma-2)}A \leq \frac{4}{2} \times \frac{6}{4}A = 3A. \tag{22}$$

Thus we obtain the next estimate.

Proposition 3. *If $B \geq 4$ and $\sigma \geq 6$, then*

$$e \leq 3A. \tag{23}$$

Next, we shall establish that even if $B \geq 3$,then $e \leq 3A$, except for certain cases.

6. Case when $e > 3A$

In that follows we assume $e > 3A$ and $\sigma \geq 6$.

By Proposition 3, we may assume that $B = 3$. Then

$$0 \leq \mathcal{Z}^* = \sigma(2 + \overline{\nu_1} - \sigma) - \overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g}.$$

Recalling that $\sigma = p + 2\overline{\nu_1} + 2$, we get

$$2 + \overline{\nu_1} - \sigma = 2 + \overline{\nu_1} - (p + 2\overline{\nu_1} + 2) = -p - \overline{\nu_1}.$$

Hence,

$$\begin{aligned}\mathcal{Z}^* &= -\sigma(p + \overline{\nu_1}) - \overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g} \\ &= \overline{\nu_1}(-\sigma - k + A_1) - \tilde{k} - p\sigma + A_1 + 2\overline{g} \\ &= \overline{\nu_1}(-\sigma - k + A) - \tilde{k} - p\sigma + A + \overline{g}(1 - \overline{\nu_1}).\end{aligned}$$

But $A-\sigma<\frac{u}{3}$ and $-\sigma-k+A<\frac{u}{3}-k=-4p-\frac{5u}{3}$. Therefore, since $A<\sigma+\frac{u}{3}=p+\nu_1+\frac{u}{3}$, it follows that

$$\begin{aligned}0\le \mathcal{Z}^* &< -\overline{\nu_1}(4p+\frac{5u}{3})-\tilde{k}-p\sigma+A+\overline{g}(1-\overline{\nu_1})\\ &< -\overline{\nu_1}(4p+\frac{5u}{3}-2)-\tilde{k}-p\sigma+\frac{u}{3}+p+2+\overline{g}(1-\overline{\nu_1}).\end{aligned}$$

Thus,

$$0\le \overline{\nu_1}(4p+\frac{5u}{3}-2)<-\tilde{k}+\frac{u}{3}+p+2-p\sigma+\overline{g}(1-\overline{\nu_1}).$$

Supposing that $p>0$, we shall derive a contradiction.
$4p+\frac{5u}{3}-2>0$ and

$$\begin{aligned}-\tilde{k}+\frac{u}{3}+p+2 &= -p(k-2p)+\frac{u}{3}+p+2\\ &= -p(2p+2u)+\frac{u}{3}+p+2\\ &= -2p^2+2+p-pu-pu+\frac{u}{3}<0,\end{aligned}$$

except for $p=1$ and $u=0$.

However, if $p=1$ and $u=0$ then $\sigma=1+2\nu_1, k=4, \tilde{k}=1, A\le 2\sigma-1=2\nu_1$. Thus,

$$\begin{aligned}\mathcal{Z}^* &= \overline{\nu_1}(-\sigma-k+A)-\tilde{k}-p\sigma+A+\overline{g}(1-\overline{\nu_1})\\ &= \overline{\nu_1}(-1-2\nu_1-4+A)-1-(1+2\nu_1)+A+\overline{g}(1-\overline{\nu_1})\\ &\le -5\overline{\nu_1}-2+\overline{g}(1-\overline{\nu_1}).\end{aligned}$$

But, if $\overline{g}\ge 0$, then

$$-5\overline{\nu_1}-2+\overline{g}(1-\overline{\nu_1})\le -5\overline{\nu_1}-2\le -12.$$

Moreover, if $\overline{g}=-1$, then

$$-5\overline{\nu_1}-2+\overline{g}(1-\overline{\nu_1})=-4\overline{\nu_1}-1\le -9.$$

$$0\le \overline{\nu_1}(4p+\frac{5u}{3}-2)<0.$$

This is a contradiction. Thus, $p=0$ has been established.

By $p = 0$, we have

$$\begin{aligned} \mathcal{Z}^* &= -\sigma(p + \overline{\nu_1}) - \overline{\nu_1}k - \tilde{k} + \nu_1 A_1 + 2\overline{g} \\ &= -2\nu_1(\overline{\nu_1}) - 2\overline{\nu_1}u + \nu_1 A_1 + 2\overline{g} \\ &= -2\overline{\nu_1}(u + \nu_1) + \nu_1 A_1 + 2\overline{g} \\ &= -2\nu_1(\overline{\nu_1} + u) + \nu_1 A_1 + 2\overline{g}. \end{aligned}$$

From the inequality (16), we have

$$\begin{aligned} \sigma A &\geq e(1 - \frac{2}{B})(\sigma - 2) + u(1 + \frac{2}{B})(\sigma - 2) + (\sigma - 4)\overline{g} \\ &= \frac{e(\sigma - 2)}{3} + \frac{5u(\sigma - 2)}{3} + (\sigma - 4)\overline{g}. \end{aligned}$$

Thus

$$3\sigma A \geq e(\sigma - 2) + 5u(\sigma - 2) + 3(\sigma - 4)\overline{g}. \tag{24}$$

Since $e > 3\sigma$ by hypothesis, it follows that

$$e\sigma > e(\sigma - 2) + 5u(\sigma - 2) + 3(\sigma - 4)\overline{g}.$$

Hence,

$$2e > 5u(\sigma - 2) + 3(\sigma - 4)\overline{g}.$$

Since $p = 0$ we obtain

$$\begin{aligned} 2e = 2(3\sigma + u) &\geq 5u(\sigma - 2) + 3(\sigma - 4)\overline{g} \\ &= 10u(\nu_1 - 1) + 6(\nu_1 - 2)\overline{g}. \end{aligned}$$

Hence,

$$2(6\nu_1 + u) \geq 10u(\nu_1 - 1) + 6(\nu_1 - 2)\overline{g}.$$

Thus,

$$12u \geq 10u\nu_1 + 6\nu_1\overline{g} - 12\overline{g} - 12\nu_1;$$
$$6u + 6\nu_1 \geq 5u\nu_1 + 3\nu_1\overline{g} - 6\overline{g}.$$

Therefore,

$$6u + 6\overline{g} \geq 5u\nu_1 + 3\nu_1\overline{g} - 6\nu_1 = (5u + 3\overline{g} - 6)\nu_1. \tag{25}$$

Thus we have the next two cases to examine, separately.

1) $5u + 3\overline{g} - 6 < 0$. Then $u = 0, 1, 2$.
Suppose that $u = 1$. Then

$$\overline{g} \leq 2 - \frac{5u}{3} = \frac{1}{3} < 1.$$

Hence, $\overline{g} = 0, -1$.

(i) $\overline{g} = 0$.
Then

$$\begin{aligned}\mathcal{Z}^* &= -2\nu_1(\overline{\nu_1} + u) + \nu_1 A_1 + 2\overline{g} + 2u \\ &= 2\nu_1(1-\nu_1) - 2(\nu_1 - 1) + \nu_1 A_1.\end{aligned}$$

Since $A \leq \sigma - 1 = 2\nu_1 - 1$, it follows that

$$2\nu_1(1-\nu_1) - 2(\nu_1 - 1) + \nu_1 A_1 \leq 2\nu_1(1-\nu_1) - 2(\nu_1 - 1) < 0.$$

Thus $\mathcal{Z}^* < 0$, a contradiction.

(ii) $\overline{g} = -1$. Then $A_1 = A + 1 \leq 2\nu_1$ and so

$$\begin{aligned}\mathcal{Z}^* &= -2\nu_1(\overline{\nu_1} + u) + \nu_1 A_1 + 2\overline{g} \\ &= -2\nu_1(2\nu_1 - 1) + \nu_1(A_1) - 2 \\ &\leq -2\nu_1(2\nu_1 - 1) + \nu_1(2\nu_1) - 2 \\ &= 0.\end{aligned}$$

Thus, $\mathcal{Z}^* = 0$. Hence,$e = 3\sigma + 1 = 6\nu_1 + 1, \widetilde{B} = 2e - 3\sigma = 3\sigma + 2$. Since $A = 2\nu_1 - 1$,it follows that $e - 3A = 4 > 0$.

$$g = 3(2\nu_1 - 1)\nu_1 - r\frac{\nu_1(\nu_1 - 1)}{2} = 0.$$

Therefore,

$$r = \frac{6(2\nu_1 - 1)}{\nu_1 - 1} = 12 + \frac{6}{\nu_1 - 1}. \tag{26}$$

From this, it follows that $\nu_1 - 1 = 2, 3, 6$ and we obtain the following types.

(1) The type is $[6 * 19, 3; 3^1 5], A = 5$.
(2) The type is $[8 * 25, 3; 4^1 4], A = 7$.
(3) The type is $[12 * 37, 3; 7^1 3], A = 11$.

Suppose that $u = 2$. Then

$$\overline{g} \leq 2 - \frac{5u}{3} = \frac{-4}{3} < 1.$$

Hence, $\overline{g} < -1$, a contradiction.
Therefore, $u = 0$.

2) $5u + 3\overline{g} - 6 \geq 0$.
Then since $\sigma = 2\nu_1 \geq 6$, it follows that $\nu_1 \geq 3$ and thus

$$6u + 6\overline{g} \geq (5u + 3\overline{g} - 6)\nu_1 \geq 3(5u + 3\overline{g} - 6). \tag{27}$$

This implies the next inequality:

$$6 \geq 3u + \overline{g} \geq 3u - 1.$$

Hence, $u \leq 2$ and $\overline{g} \leq 6 - 3u$.

Moreover,

$$\begin{aligned} \mathcal{Z}^* &= -2\nu_1(\overline{\nu_1} + u) + \nu_1 A_1 + 2\overline{g} + 2u \\ &= -\nu_1(2(\overline{\nu_1} + u) - A_1) + 2\overline{g} + 2u. \end{aligned}$$

Then $A_1 = A - \overline{g} \leq 2\nu_1 - 1 - \overline{g}$ and so

$$2(\overline{\nu_1} + u) - A_1 \geq 2(\overline{\nu_1} + u) - 2\nu_1 + 1 + \overline{g} = 2u - 1 + \overline{g}.$$

Hence,

$$\begin{aligned} \mathcal{Z}^* &= -\nu_1(2(\overline{\nu_1} + u) - A_1) + 2\overline{g} + 2u \\ &\leq -\nu_1(2u - 1 + \overline{g}) + 2\overline{g} + 2u \\ &= -\nu_1(2u - 1) + 2u - \nu_1\overline{g} + 2\overline{g} \\ &< 0. \end{aligned}$$

This is absurd. Thus, $k = 0$ is proved.

7. CASE WHEN $k = 0$

$\varepsilon = 2\nu_1 - 1 - A$ satisfies $\varepsilon \geq 0$, since $A \leq \sigma - 1 = 2\nu_1 - 1$. From

$$0 \leq \mathcal{Z}^* = \nu_1(2 + A - 2\nu_1 - \overline{g}) + 2\overline{g}, \tag{28}$$

it follows that

$$\begin{aligned} \mathcal{Z}^* &= \nu_1(1 - \varepsilon - \overline{g}) + 2\overline{g} \\ &= \nu_1(1 - \varepsilon) + \overline{g}(2 - \nu_1). \end{aligned}$$

If $\overline{g} \geq 0$, then

$$\mathcal{Z}^* = \nu_1(1 - \varepsilon) + \overline{g}(2 - \nu_1) \leq \nu_1(1 - \varepsilon).$$

Hence, $\varepsilon \leq 1$.

Otherwise,

$$\mathcal{Z}^* = \nu_1(1 - \varepsilon) - (2 - \nu_1) = \nu_1(2 - \varepsilon) - 2.$$

Hence, $\varepsilon \leq 1$.

Thus we have the following cases:

(1) $\varepsilon = 0, \mathcal{Z}^* = \nu_1 + \overline{g}(2 - \nu_1)$
   - (a) $\overline{g} = -1, \mathcal{Z}^* = 2\nu_1 - 2$,
   - (b) $\overline{g} = 0, \mathcal{Z}^* = \nu_1$,
   - (c) $\overline{g} = 1, \mathcal{Z}^* = 2$,

(d) $\overline{g} = 2, \mathcal{Z}^* = 4 - \nu_1$,
(e) $\overline{g} = 3, \mathcal{Z}^* = 2(3 - \nu_1)$.
(2) $\varepsilon = 1, \mathcal{Z}^* = \overline{g}(2 - \nu_1)$
(a) $\overline{g} = -1, \mathcal{Z}^* = \nu_1 - 2$,
(b) $\overline{g} = 0, \mathcal{Z}^* = 0$,
(c) $\overline{g} = 1, \mathcal{Z}^* = 2 - \nu_1$.

7.1. **case when** $\varepsilon = 0, \overline{g} = -1$.
In this case, $A = 2\nu_1 - 1, g = 0$.

If $\nu_1 = 3$,then $\mathcal{Z}^* = t_2$ and so $t_2 = 4$. The type is $[6 * 18, 3; 3^{t_3}, 2^4]$.
$g = 5{\cdot}8 - 3t_3 - 4 = 0$. Hence, $t_3 = 12$ and the type becomes $[6{*}18, 3; 3^{12}, 2^4]$ and $Z^2 = 4$.

If $\nu_1 = 4$,then $\mathcal{Z}^* = 2(t_2 + t_3)$ and so $t_2 + t_3 = 3$. The type is $[8 * 24, 3; 4^{t_4}, 3^{t_3}, 2^{t_2}]$.
$g = 7 \cdot 11 - 6t_4 - 3t_3 - t_2 = 0$. Hence, $37 = 3t_4 + t_3$.Then $t_3 = 1, t_4 = 12$ and the type becomes $[8 * 24, 3; 4^{12}, 3, 2^2]$. $Z^2 = 6$.

If $\nu_1 > 4$,then $\mathcal{Z}^* > \nu_1 - 2 + 2(\nu_1 - 3)$. From $\mathcal{Z}^* = 2\nu_1 - 2$, it follows that $\nu_1 \leq 6$. Hence,$\nu_1 = 5$ or 6.

If $\nu_1 = 5$,then $\mathcal{Z}^* = 3(t_2 + t_4) + 4t_5$ and so $3(t_2 + t_4) + 4t_3 = 8$. Then $t_3 = 2$ and the type is $[10 * 30, 3; 5^{t_5}, 3^2]$. $g = 126 - (10t_5 + 6) = 0$. From this it follows that $t_5 = 12$ and the type becomes $[10 * 30, 3; 5^{12}, 3^2]$.

If $\nu_1 = 6$,then $\mathcal{Z}^* = 4(t_2 + t_5) + 6(t_3 + t_4) = 10$ and so $t_2 + t_5 = t_3 + t_4 = 1$. Then the type is $[12 * 36, 3; 6^{t_6}, 5^{t_5}, 4^{t_4}, 3^{t_3}, 2^{t_2}]$.

- $g = 187 - (15t_6 + 10t_5 + 6t_4 + 3t_3 + t_2) = 0$,
- $t_2 + t_5 = t_3 + t_4 = 1$.

These imply $t_2 = 1, t_4 = 1, t_6 = 12$. The type is $[12 * 36, 3; 6^{12}, 4, 2]$.

7.2. **case when** $\varepsilon = 0, \overline{g} = 0$.
In this case, $A = 2\nu_1 - 1, g = 1$ and $\mathcal{Z}^* = \nu_1$. Then $\nu_1 = \mathcal{Z}^* \geq 2(\nu - 3)$. Hence, $\nu_1 \leq 6$.
We have the four cases to examine, separately.
(1) $\nu_1 = 3$. Then $\mathcal{Z}^* = t_2 = 3$. The type is $[6 * 18, 3; 3^{t_3}, 2^3]$.
$g = 5{\cdot}8 - 3t_3 - 3 = 1$. Hence, $t_3 = 12$ and the type becomes $[6{*}18, 3; 3^{12}, 2^3]$ and $Z^2 = 5$.
(2) $\nu_1 = 4$. Then $\mathcal{Z}^* = 2(t_2 + t_3) = 4$ and so $t_2 + t_3 = 2$. The type is $[8 * 24, 3; 4^{t_4}, 3^{t_3}, 2^{t_2}]$.
$g = 7{\cdot}11 - 6t_4 - 3t_3 - t_2 = 1$. Hence, $37 = 3t_4 + t_3$. Then $t_3 = 1, t_4 = 12$ and the type becomes $[8{*}24, 3; 4^{12}, 3, 2]$. $Z^2 = 6{\cdot}20 - 9{\cdot}12 - 4 - 1 = 7, A = Z^2 = 7$.
(3) $\nu_1 = 5$. Then $\mathcal{Z}^* = 3(t_2 + t_4) + 4t_3 = 4$ and so $t_3 = 1, t_2 = t_4 = 0$. The type is $[10 * 30, 3; 5^{t_5}, 3]$.

$g = 9 \cdot 14 - 10t_5 - 3 = 1$, which has no solution.

(4) $\nu_1 = 6$. Then $\mathcal{Z}^* = 4(t_2 + t_5) + 6(t_3 + t_4) = 6$ and so $t_3 + t_4 = 1, t_2 = t_5 = 0$. The type is $[12 * 36, 3; 6^{t_6}, 4^{t_4}, 3^{t_3}]$.

- $g = 187 - (6t_4 + 3t_3) = 1$,
- $t_3 + t_4 = 1$.

These imply $t_4 = 1, t_6 = 12$. The type is $[12 * 36, 3; 6^{12}, 4]$.

7.3. **case when** $\varepsilon = 0, \overline{g} = 1$.

In this case, $A = 2\nu_1 - 1, g = 2$ and $\mathcal{Z}^* = 2$. From $\mathcal{Z}^* \geq nu_1 - 2$, it follows that $\nu_1 = 3$ or 4.

If $\nu_1 = 3$,then $\mathcal{Z}^* = t_2 = 2$. The type is $[6 * 18, 3; 3^{t_3}, 2^2]$.

$g = 5 \cdot 8 - 3t_3 - 3 = 2$. Hence, $t_3 = 12$ and the type becomes $[6*18, 3; 3^{12}, 2^2]$ and $Z^2 = 5$.

If $\nu_1 = 4$,then $\mathcal{Z}^* = 2(t_2 + t_3) = 2$ and so $t_2 + t_3 = 1$. The type is $[8 * 24, 3; 4^t_4, 3^{t_3}, 2^{t_2}]$.

$g = 7 \cdot 11 - 6t_4 - 3t_3 - t_2 = 2$. Hence, $37 = 3t_4 + t_3$.Then $t_3 = 1, t_4 = 12$ and the type becomes $[8 * 24, 3; 4^{12}, 3]$

7.4. **case when** $\varepsilon = 0, \overline{g} = 2$.

In this case, $A = 2\nu_1 - 1, g = 3$ and $\mathcal{Z}^* = 4 - \nu_1$.

If $4 \neq \nu_1$ then $4 - \nu_1 \geq \nu_1 - 2$; thus $\nu_1 = 3$.

If $\nu_1 = 4$,then $\mathcal{Z}^* = 0$ and so $t_3 = 0$. The type is $[8 * 24, 3; 4^{t_4}]$. $g = 7 \cdot 11 - 6t_4 = 3$, which has no solution.

If $\nu_1 = 3$,then $\mathcal{Z}^* = 1$. Then $t_2 = 1$ and The type is $[6 * 18, 3; 3^{t_3}, 2]$.

$g = 5 \cdot 8 - 3t_3 - 3 = 3$. Hence, $t_3 = 12$ and the type becomes $[6*18, 3; 3^{12}, 2]$ and $Z^2 = 6$.

7.5. **case when** $\varepsilon = 0, \overline{g} = 3$.

In this case, $A = 2\nu_1 - 1, g = 4$ and $\mathcal{Z}^* = 2(3 - \nu_1)$.

Then $\nu_1 = 3$ and the type becomes $[6 * 18, 3; 3^{12}]$ and $Z^2 = 7$.

7.6. **case when** $\varepsilon = 1, \overline{g} = -1$.

In this case, $A = 2\nu_1 - 2, g = 0$ and $\mathcal{Z}^* = \nu_1 - 2$.

Since $\mathcal{Z}^* = \nu_1 - 2$, it follows that $t_2 + t_{\nu_1 - 1} = 1$. Note that

$$g = (2\nu_1 - 1)(3\nu_1 - 1) - \frac{\nu_1(\nu_1 - 1)}{2} t_{\nu_1} - t_2 - \frac{(\nu_1 - 2)(\nu_1 - 1)}{2} t_{\nu_1 - 1} = 0.$$

If $t_2 = 1$ then $t_{\nu_1 - 1} = 0$ and hence,

$$12\nu_1 - 10 - (\nu_1 - 1)t_{\nu_1 - 1} = 0.$$

Thus,

$$t_{\nu_1 - 1} = \frac{12\nu_1 - 10}{\nu_1 - 1} = 12 + \frac{2}{\nu_1 - 1}.$$

Therefore, $\nu_1 = 3$ and $t_3 = 13$. $g = 5 \cdot 8 - 3 \times 13 - t_2 = 0$. Hence, $t_2 = 1$ and the type becomes $[6*18, 3; 3^{12}, 2]$ and $Z^2 = 3$. Note that $A = 3+1 = 4 = 2\nu_1 - 2$.

If $t_2 = 0$ then $t_{\nu_1 - 1} = 1$ and hence,

$$g = (2\nu_1 - 1)(3\nu_1 - 1) - \frac{\nu_1(\nu_1 - 1)}{2} t_{\nu_1} - \frac{(\nu_1 - 2)(\nu_1 - 1)}{2} = 0.$$

$$11\nu_1 - 7 - (\nu_1 - 1)t_{\nu_1 - 1} = 0.$$

Thus,

$$t_{\nu_1 - 1} = \frac{11\nu_1 - 7}{\nu_1 - 1} = 11 + \frac{4}{\nu_1 - 1}.$$

Therefore, $\nu_1 = 5$ and $t_5 = 12, t_4 = 1$.

If $\nu_1 = 5$,then $g = 126 - (10t_5 + 6) = 0$. The type becomes $[10*30, 3; 5^{12}, 4]$.

7.7. **case when** $\varepsilon = 1, \overline{g} = 0$.

In this case, $A = 2\nu_1 - 2, g = 1$ and $\mathcal{Z}^* = 0$.

It follows that

$$g = (2\nu_1 - 1)(3\nu_1 - 1) - \frac{\nu_1(\nu_1 - 1)}{2} t_{\nu_1} = 1.$$

Hence,

$$12\nu_1 - 10 - (\nu_1 - 1)t_{\nu_1 - 1} = 0.$$

Thus,

$$t_{\nu_1 - 1} = \frac{12\nu_1 - 10}{\nu_1 - 1} = 12 + \frac{3}{\nu_1 - 1}.$$

Therefore, $\nu_1 = 4$ and $t_3 = 0$. The type is $[8 * 24, 3; 4^{t_4}]$. $g = 7 \cdot 11 - 6t_4 = 1$, which has no solution.

7.8. **case when** $\varepsilon = 1, \overline{g} = -1$.

But this case does not occur.

8. ESTIMATE OF D^2

Assume that $\sigma \geq 4$.

Supposing $B < 3$, we have two cases:

(1) If $B \neq 1$ or $B = 1$ and $3\sigma - 2e \leq 0$ then $(\sigma K_S + 2D) \cdot D \geq 0$. Hence,

$$(\sigma K_S + 2D) \cdot D = (\sigma Z - (\sigma - 2)D) \cdot D = 2\sigma\overline{g} - (\sigma - 2)D^2 \geq 0. \tag{29}$$

(2) Otherwise, $B = 1$ and $3\sigma - 2e > 0$. In this case, the type of the pairs coincides with that of the pair of plane type $[e; \nu_0, \nu_1, \cdots, \nu_r]$ where $\nu_0 = e - \sigma$. Then $2e - 3\sigma = 3\nu_0 - e$, and $(eK_0 + 3C) \cdot C = (e - 3\nu_0)\nu_0 \geq \nu_0 = e - \sigma$. Thus

$$(eK_S + 3D) \cdot D = \sum_{j=0}^{r} (e - 3\nu_j)\nu_j \geq \nu_0 = e - \sigma = u + \nu_1. \tag{30}$$

Hence,

$$(eK_S+3D)\cdot D = (eZ-(e-3)D)\cdot D = 2e\overline{g}-(e-3)D^2 \geq e-\sigma = u+\nu_1 > 0. \tag{31}$$

Thus we have two cases: If $B \neq 1$ or $B = 1$ and $3\sigma - 2e \geq 0$ then

$$2\sigma\overline{g} - (\sigma-2)D^2 \geq 0. \tag{32}$$

In other words,

$$D^2 \leq \frac{2\sigma}{\sigma-2}\overline{g}. \tag{33}$$

Otherwise,

$$D^2 \leq \frac{2e}{e-3}\overline{g}. \tag{34}$$

Therefore, under the assumption that $g > 0$, if $\sigma \geq 4$ or if $e \geq 6$ then

$$D^2 \leq 4\overline{g}. \tag{35}$$

Moreover, if $\sigma \geq 5$ or if $e \geq 7$ then

$$D^2 \leq \frac{10}{3}\overline{g} \quad \text{or } D^2 \leq \frac{7}{2}\overline{g}. \tag{36}$$

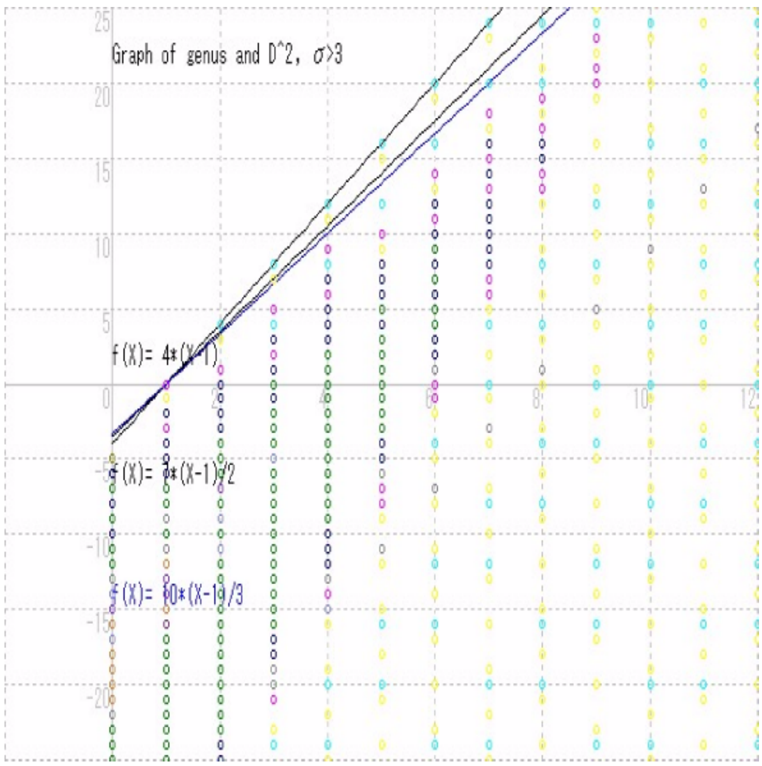

FIGURE 1. D^2 and genus

8.1. **case when** $\sigma = 4$**.** We shall enumerate types which satisfy $\sigma = 4$ and $D^2 > \frac{10}{3}\overline{g}$.

We distinguish the various cases according to B.

1) $B = 0$. Then $e = \sigma + u = 4 + u$ and so $D^2 = 32 + 8u - 4r, \overline{g} = 8 + 3u - r$,$r$ being the number of singular points. By

$$D^2 - \frac{10}{3}\overline{g} = 32 + 8u - 4r - \frac{10}{3}(8 + 3u - r) = 2\frac{8 - 3u - r}{3} > 0$$

we have the following cases:

- $u = 0$. Then $r \leq 7$.
- $u = 1$. Then $r \leq 4$.
- $u = 2$. Then $r \leq 1$.

TABLE 1

u	0			1			2		
r	D^2	$\overline{g}$	$D^2/\overline{g}$	D^2	$\overline{g}$	$D^2/\overline{g}$	D^2	$\overline{g}$	$D^2/\overline{g}$
0	32	8	4	40	11	3.64	48	14	3.483
1	28	7	4	36	10	3.6	44	13	3.38
2	24	6	4	32	9	3.56	40	12	3.33
3	20	5	4	28	8	3.5	36	11	3.27
4	16	4	4	24	7	3.43	32	10	3.2
5	12	3	4	20	6	3.33	28	9	3.111

2) $B = 1$. Then $e = \sigma + u + 2 = 6 + u$ and so $D^2 = 32 + 8u - 4r, \overline{g} = 8 + 3u - r$,$r$ being the number of singular points. Thus we have the same result as above. Needless to say , in the case when $B = 2$, the same result is obtained.

3) The type is plane type such that $[e; 2^r]$.

Then $D^2 = e^2 - 4r, \overline{g} = \frac{e(e-3)}{2} - r$,$r$ being the number of singular points.

Thus, $4\overline{g} - D^2 = e(e - 6)$.

In particular,if $e = 6$ then $4\overline{g} - D^2 = 0$.

If $e = 7$ then $7 = e > 3\nu_0$. Hence,$\nu_0 = 2$. Therefore, pairs of the type $[7; 2^r]$ have the following invariants: $D^2 = 49 - 4r, \overline{g} = 14 - r$.

Hence, $4\overline{g} - D^2 = e(e - 6) = 7$.

Moreover,

$$\frac{10}{3}\overline{g} - D^2 = \frac{2r - 7}{3}. \tag{37}$$

Hence, if $\frac{10}{3}\overline{g} - D^2 < 0$ then $r \leq 3$.

Theorem 1. *Suppose that* $\sigma \geq 4$ *or* $e \geq 6$*. Then*

$$D^2 \leq 4\overline{g}.$$

Moreover,

$$D^2 \leq \frac{10}{3}\overline{g}$$

except for the following cases.

(1) *The types are* $[4*(6+u);2^r]^*$ *where* $r+3u<8$.
(2) *The types are* $[6;2^r]$ *where* $r \leq 8$.
(3) *The types are* $[7;2^r]$ *where* $r \leq 3$.

9. ESTIMATE OF Z^2

Supposing that $\sigma \geq 4$ or $e \geq 6$, we get $Z^2 \geq \frac{2(\sigma-2)}{\sigma}\overline{g}$ or $Z^2 \geq \frac{2(e-3)}{e}\overline{g}$
If $\sigma = 4$ or $e = 6$ then $A = Z^2 - \overline{g} = 0$.
If $\sigma \geq 5$, then $Z^2 \geq \frac{6}{5}\overline{g}$.
Moreover, if $e \geq 7$, then $Z^2 \geq \frac{8}{7}\overline{g}$. Furthermore, if $e \geq 8$, then $Z^2 \geq \frac{5}{4}\overline{g}$.

TABLE 2

u	0			1			2		
r	Z^2	$\overline{g}$	$Z^2/\overline{g}$	Z^2	$\overline{g}$	$Z^2/\overline{g}$	Z^2	$\overline{g}$	$Z^2/\overline{g}$
0	8	8	1	12	11	1.09	16	14	1.143
1	7	7	1	11	10	1.1	15	13	1.154
2	6	6	1	10	9	1.11	14	12	1.17
3	5	5	1	9	8	1.13	13	11	1.19
4	4	4	1	8	7	1.143	12	10	1.2
5	3	3	1	7	6	1.17	11	9	1.22
6	2	2	1	6	5	1.2	10	8	1.25

Theorem 2. *Suppose that* $\sigma \geq 4$ *or* $e \geq 6$. *Then*

$$A = Z^2 - \overline{g} \geq 0.$$

$$Z^2 \geq \frac{6}{5}\overline{g}$$

except for the following cases.

(1) *The types are* $[4*(6+u);2^r]^*$ *where* $r+2u<8$.
(2) *The types are* $[6;2^r]$ *where* $r \leq 8$.
(3) *The types are* $[7;2^r]$ *where* $r \leq 3$.

Note that the point under the line $y = x$ corresponds to the nonsigular quintic which has the invariant $Z^2 = 4$,genus $= 6$:

10. GRAPH OF $(\alpha\ ,\omega)$

Under the assumption $\sigma \geq 4$ or $e \geq 6$, we have

- $D^2 \leq \frac{2\sigma}{\sigma-2}\overline{g}$.
- $D^2 \leq \frac{2e}{e-3}\overline{g}$.

FIGURE 2. Z^2 and genus

Thus

$$\omega = 3\overline{g} - D^2 \geq 3\overline{g} - \frac{2\sigma}{\sigma - 2}\overline{g} = \frac{\sigma - 6}{\sigma - 2}\overline{g}.$$

From $\alpha = \omega + \overline{g}$, it follows that

$$\omega \geq \frac{\sigma - 6}{\sigma - 2}(\alpha - \omega).$$

Hence,

$$\omega \geq \frac{\sigma - 6}{2\sigma - 4}\alpha. \tag{38}$$

Similarly, we obtain

$$\omega \geq \frac{e - 9}{2e - 12}\alpha. \tag{39}$$

Suppose that $\sigma \geq 7$ or $e \geq 10$. Then

$$\omega \geq \frac{\alpha}{6} \quad \text{or} \quad \omega \geq \frac{\alpha}{8}.$$

If $e = 9$ then the type is $[9; 1]$ or $[9; 2^{r+1}]$. In the latter case, the type is rewritten as $[7 * 9; 2^r]$.

It is easy to compute the invariants:

$$\overline{g} = 27 - (r+1) > 0, \omega = r+1, \alpha = 27.$$

Then

$$\frac{\omega}{\alpha} = \frac{r+1}{27} = \frac{1}{27}, \frac{2}{27}, \frac{3}{27} = \frac{1}{9},$$

If the plane type is $[e; 3^{t_3}, 2^{t_2}]$, then

$$\overline{g} = \frac{e(e-3)}{2} - 3t_3 - t_2 \geq 0, \omega = \frac{e(e-9)}{2} + t_2, \alpha = e(e-6) - 3t_3.$$

Thus for e=10, we get

$$\overline{g} = 35 - 3t_3 - t_2 \geq 0, \omega = 5 + t_2, \alpha = 40 - -3t_3.$$

Hence, $6\omega - \alpha < 0$ implies that $2t_2 + t_3 < \frac{10}{3}$.

TABLE 3

t_2	0	0	0	1	1	1	2	2	2	3	3	3
t_3	ω	α	ω/α	ω	α	ω/α	ω	α	ω/α	ω	α	ω/α
0	5	40	0.125	6	40	0.15	7	40	0.175	8	40	0.2
1	5	37	0.135	6	37	0.162	7	37	0.189	8	37	0.216
2	5	34	0.147	6	34	0.176	7	34	0.206	8	34	0.235
3	5	31	0.161	6	31	0.194	7	31	0.226	8	31	0.258
4	5	28	0.179	6	28	0.214	7	28	0.25	8	28	0.286

Thus we obtain the following result:

Theorem 3. *If $\sigma \geq 7$, then $\frac{\omega}{\alpha} \geq \frac{1}{6}$ except for the cases when 1) the type is $[9; 1]$ or 2) $[9; 2^{r+1}]$ where $r \leq 3$ or 3) $[10; 3^{t_3}, 2^{t_2}]$, where $2t_2 + t_3 \leq 3$.*