Super perfect numbers and Mersenne perfect numbers

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March 7, 2019

1 perfect numbers

6 is called a perfect number because divisors of 5 are 1,2,3,6 and the sum of these (excluding 6) is $1 + 2 + 3 = 6$.

By $\sigma(a)$ we denote the sum of divisors of a positive integer a. If $\sigma(a) = 2a$, then a is a perfect number by definition.

If $a = 2^e$, $\sigma(a) = 1 + \cdots + 2^e = 2^{e+1} - 1$. We assume that $p = 2^{e+1} - 1$ is prime, and let α be ap. Then $\sigma(\alpha) = \sigma(a)\sigma(p) = p(p + 1) = p(2^{e+1}) = 2p2^e = 2\alpha$.

Thus α are perfect numbers due to Euclid.

2 translation

Given an integer m, we shall study positive integers a satisfying $\sigma(a)-2a = -m$.

If $a = 2^e q(q$: prime), then such a solution is said to be of type A.

Assuming $a = 2^e q$, we get $\sigma(a) = \sigma(2^e q) = \sigma(2^e)\sigma(q)$.

Making use of $N = 2^{e+1} - 1$, $\sigma(q) = q + 1$ we get $\sigma(a) = \sigma(2^e)\sigma(q) = N(q + 1) = Nq + N$.

From $Nq = (2^{e+1} - 1)q = 2a - q$ it follows that $\sigma(a) = Nq + N = 2a - q + N$.

By $\sigma(a) - 2a = -q + N$ and $\sigma(a) - 2a = -m$, we obtain , $-m = -q + N$. Hence,$q = N + m = 2^{e+1} - 1 + m$.

Conversely, given integer m(This is a translation parameter) , if $2^{e+1} - 1 + m$ is prime, then $a = 2^e q$ satisfies $\sigma(a) - 2a = -m$.

$a = 2^e q$ is said to be a perfect number with translation (parameter) m in strict sense.

In general ,if a positive integer a satisfies $\sigma(a) - 2a = -m$, then we say that a is a perfect number with translation parameter m in broard sense.

Table 1: perfect numbers with translation $m = -12$

a	factor
$m = -12$	
24	$2^3 * 3$
54	$2 * 3^3$
304	$2^4 * 19$
30	$2 * 3 * 5$
42	$2 * 3 * 7$
66	$2 * 3 * 11$
78	$2 * 3 * 13$
102	$2 * 3 * 17$
114	$2 * 3 * 19$
138	$2 * 3 * 23$
174	$2 * 3 * 29$
186	$2 * 3 * 31$
222	$2 * 3 * 37$
246	$2 * 3 * 41$
258	$2 * 3 * 43$
282	$2 * 3 * 47$
318	$2 * 3 * 53$
354	$2 * 3 * 59$

$a = p(p > 3$ primes) are solutions, that are called solutions of type B.

Table 2: perfect numbers with translation $m = -10, -8, -6$

a	factor	
$m = -10$		type A
40	$2^3 * 5$	
1696	$2^5 * 53$	
518656	$2^9 * 1013$	
$m = -8$		type A
56	$2^3 * 7$	
368	$2^4 * 23$	
128768	$2^8 * 503$	
836	$2^2 * 11 * 19$	type D
11096	$2^3 * 19 * 73$	
17816	$2^3 * 17 * 131$	
77744	$2^4 * 43 * 113$	
45356	$2^2 * 17 * 23 * 29$	
91388	$2^2 * 11 * 31 * 67$	
254012	$2^2 * 11 * 23 * 251$	
388076	$2^2 * 13 * 17 * 439$	
$m = -6$		
8925	$3 * 5^2 * 7 * 17$	
32445	$3^2 * 5 * 7 * 103$	
442365	$3 * 5 * 7 * 11 * 383$	

$2^3 * 5, 2^5 * 53, (2^e q)$ are said to be solutions of type A
$2^2 * 11 * 19, 2^3 * 19 * 73, (2^e qr)$ are said to be solutions of type D

Table 3: perfect numbers with translation $m = -4, -2, 0$

a	factor	
$m = -4$		type A
12	$2^2 * 3$	
88	$2^3 * 11$	
1888	$2^5 * 59$	
32128	$2^7 * 251$	
521728	$2^9 * 1019$	
70	$2 * 5 * 7$	
4030	$2 * 5 * 13 * 31$	
5830	$2 * 5 * 11 * 53$	
$m = -2$		type A
20	$2^2 * 5$	
104	$2^3 * 13$	
464	$2^4 * 29$	
1952	$2^5 * 61$	
130304	$2^8 * 509$	
522752	$2^9 * 1021$	
650	$2 * 5^2 * 13$	
$m = 0$	perfect numbers	type A
6	$2 * 3$	
28	$2^2 * 7$	
496	$2^4 * 31$	
8128	$2^6 * 127$	

Table 4: perfect numbers with translation $m = 2, 4$

a	factor	
$m = 2$		
3	3	
10	$2 * 5$	type A
136	$2^3 * 17$	
32896	$2^7 * 257$	
$m = 4$		type A
5	5	
14	$2 * 7$	
44	$2^2 * 11$	
152	$2^3 * 19$	
2144	$2^5 * 67$	
8384	$2^6 * 131$	
110	$2 * 5 * 11$	type D
884	$2^2 * 13 * 17$	
18632	$2^3 * 17 * 137$	
116624	$2^4 * 37 * 197$	
$m = 6$		
7	7	
15	$3 * 5$	
592	$2^4 * 37$	type A
52	$2^2 * 13$	
315	$3^2 * 5 * 7$	
1155	$3 * 5 * 7 * 11$	

3 super perfect number with translation m

If α is a perfect number , then it is written as $2^e q$.

We shall study 2^e and q, independently.

In general, If $q = 2^{e+1} - 1 + m$ is prime then $a = 2^e$ satisfies the following equalities.

By $2^{e+1} - 1 = \sigma(a)$, $A = \sigma(a) + m$ is prime, hence $\sigma(A) = A + 1 = 2a + m$.

If a positive integer a and A satisfy $A = \sigma(a) + m$, $\sigma(A) = 2a + m$, then a is callded a super perfect number with translation m.

A is said to be a partner of a. $B = \sigma(A) - 1$ is called a shadow.

Super perfect number was introduced by D.Suryanaryana when $m = 0$, in 1969. He proves if a super perfect number a is even , then a turns out to be 2^e such that $p = 2^{e+1} - 1$ is prime (Mersenne prime).

Proposition 1 *If a is 2^e for some $e > 0$, then A is prime. The converse is true. (Under the following hypothesis.)*

Proof.

By definition $A = \sigma(a) + m$ and $\sigma(A) = 2a + m$.

Subtracting these

$$A - \sigma(A) = \sigma(a) - 2a.$$

If a is 2^e then $\sigma(a) - 2a = -1$. Hence,$A - \sigma(A) = -1$. Thus , A is prime.

Conversely, if A is prime, then $\sigma(A) = A + 1$. Thus $\sigma(a) - 2a = -1$. Hence, $a = 2^e$ by making use of the the following hypothesis.

Fact 1 *If $\sigma(a) - 2a = -1$ then a is said to be an almost perfect number. This should to be 2^e for some e. This is called conjecture of almost perfect numbers.*

End.

If $a = 3p, (p > 3$: primes) then partner $A = 2q$ and both primes p, q satisfy $q = 2p - 7$.

In general if odd primes p, q satisfy $q = ap + b$, where integers $a > 0, b$ have the properties (1) $p + q \equiv 1 \bmod 4$, .(2) a, b are relatively primes, then p, q are said to be super twin primes by Hiroto Takahashi.

His conjecture stating that given integers a, b, there exist infinitely many super twin primes was presented in Japanese Mathematical Education Society Meeting 2018 at Makuhari,Chiba .

Since $m = -14, a = p$, $A = \sigma(a) - 14 = p - 13 = 6q$, $\sigma(A) = 2a + m = 2(p - 7)$. Thus, $p = 6q + 13$. So $(q, 6q + 13)$ are super twin primes.

Table 5: super perfect numbers with translation $m = -18$

a	factor	A	factor	B	factor
$m = -18$					
27	3^3	22	$2 * 11$	35	$5 * 7$
16	2^4	13	13	13	13
64	2^6	109	109	109	109
1024	2^{10}	2029	2029	2029	2029
15	$3 * 5$	6	$2 * 3$	11	11
21	$3 * 7$	14	$2 * 7$	23	23
39	$3 * 13$	38	$2 * 19$	59	59
57	$3 * 19$	62	$2 * 31$	95	$5 * 19$
111	$3 * 37$	134	$2 * 67$	203	$7 * 29$
129	$3 * 43$	158	$2 * 79$	239	239
201	$3 * 67$	254	$2 * 127$	383	383
219	$3 * 73$	278	$2 * 139$	419	419
237	$3 * 79$	302	$2 * 151$	455	$5 * 7 * 13$
309	$3 * 103$	398	$2 * 199$	599	599
327	$3 * 109$	422	$2 * 211$	635	$5 * 127$
417	$3 * 139$	542	$2 * 271$	815	$5 * 163$
471	$3 * 157$	614	$2 * 307$	923	$13 * 71$
579	$3 * 193$	758	$2 * 379$	1139	$17 * 67$
669	$3 * 223$	878	$2 * 439$	1319	1319
831	$3 * 277$	1094	$2 * 547$	1643	$31 * 53$
921	$3 * 307$	1214	$2 * 607$	1823	1823
939	$3 * 313$	1238	$2 * 619$	1859	$11 * 13^2$

Table 6: super perfect numbers with translation $m = -14$

a	factor	A	factor	B	factor
$m = -14$					
16	2^4	17	17	17	17
64	2^6	113	113	113	113
128	2^7	241	241	241	241
512	2^9	1009	1009	1009	1009
247	$13 * 19$	266	$2 * 7 * 19$	479	479
37	37	24	$2^3 * 3$	59	59
67	67	54	$2 * 3^3$	119	$7 * 17$
317	317	304	$2^4 * 19$	619	619
	$a = p$		$A = 2 * 3 * q$		$p = 6q + 13$
43	43	30	$2 * 3 * 5$	71	71
79	79	66	$2 * 3 * 11$	143	$11 * 13$
127	127	114	$2 * 3 * 19$	239	239
151	151	138	$2 * 3 * 23$	287	$7 * 41$
199	199	186	$2 * 3 * 31$	383	383
271	271	258	$2 * 3 * 43$	527	$17 * 31$
331	331	318	$2 * 3 * 53$	647	647
367	367	354	$2 * 3 * 59$	719	719
379	379	366	$2 * 3 * 61$	743	743
439	439	426	$2 * 3 * 71$	863	863
487	487	474	$2 * 3 * 79$	959	$7 * 137$

Here, we find that there exist solutions $a = p$ such that$A = 2^e * q$.

By equation $A = \sigma(a) + m = p - 3 = 2^e q$, $\sigma(A) = 2a + m = 2p - 4$. Hence, $p = 2^e q + 3$. From $A = 2^e q$ it follows that $\sigma(A) = (2^{e+1} - 1)(q+1) = 2^{e+1}q - (q+1) + 2^{e+1} = 2p - 4 = 2^{e+1}q + 2$.

Thus $-(q+1) + 2^{e+1} = 2$. Hence, $q = 2^{e+1} - 3$.

Conversely, if $q = 2^{e+1} - 3$ is prime, then $a = p$ turns out to be a solution,i.e. it is a super perfect number $m = -4$.

Table 7: super perfect number with translation $m = -10$

a	factor	A	factor	B	factor
$m = -10$					
8	2^3	5	5	5	5
32	2^5	53	53	53	53
512	2^9	1013	1013	1013	1013
81	3^4	111	$3 * 37$	151	151
35	$5 * 7$	38	$2 * 19$	59	59
329	$7 * 47$	374	$2 * 11 * 17$	647	647
437	$19 * 23$	470	$2 * 5 * 47$	863	863
749	$7 * 107$	854	$2 * 7 * 61$	1487	1487
$m = -8$					
8	2^3	7	7	7	7
16	2^4	23	23	23	23
256	2^8	503	503	503	503
1024	2^{10}	2039	2039	2039	2039
221	$13 * 17$	244	$2^2 * 61$	433	433
$m = -7$					
4	2^2	0	0	0	0
$m = -6$					
17	17	12	$2^2 * 3$	27	3^3
$m = -5$					
4	2^2	2	2	2	2

Table 8: super perfect number with translation $m = -4$

a	factor	A	factor	B	factor
$m = -4$					
4	2^2	3	3	3	3
8	2^3	11	11	11	11
32	2^5	59	59	59	59
128	2^7	251	251	251	251
512	2^9	1019	1019	1019	1019
2048	2^{11}	4091	4091	4091	4091
	$a = p$		$A = 2^e * q$		
23	23	20	$2^2 * 5$	41	41
107	107	104	$2^3 * 13$	209	$11 * 19$
467	467	464	$2^4 * 29$	929	929
653	653	650	$2 * 5^2 * 13$	1301	1301
242	$2 * 11^2$	395	$5 * 79$	479	479
3077	$17 * 181$	3272	$2^3 * 409$	6149	$11 * 13 * 43$
6728	$2^3 * 29^2$	13061	$37 * 353$	13451	13451
9953	$37 * 269$	10256	$2^4 * 641$	19901	$7 * 2843$

Table 9: super perfect numbers with translation $m = -3, -2$

a	factor	A	factor	B	factor
$m = -3$					
2	2	0	0	0	0
$m = -2$					
7	7	6	$2 * 3$	11	11
4	2^2	5	5	5	5
8	2^3	13	13	13	13
16	2^4	29	29	29	29
32	2^5	61	61	61	61
256	2^8	509	509	509	509
512	2^9	1021	1021	1021	1021
2048	2^{11}	4093	4093	4093	4093
8192	2^{13}	16381	16381	16381	16381
29	29	28	$2^2 * 7$	55	$5 * 11$
253	$11 * 23$	286	$2 * 11 * 13$	503	503
889	$7 * 127$	1022	$2 * 7 * 73$	1775	$5^2 * 71$

Table 10: super perfect numbers with translation $m = -1, 0, 1, 2$

a	factor	A	factor	B	factor
$m = -1$					
2	2	2	2	2	2
$m = 0$					
2	2	3	3	3	3
4	2^2	7	7	7	7
16	2^4	31	31	31	31
64	2^6	127	127	127	127
4096	2^{12}	8191	8191	8191	8191
$m = 1$					
15	$3 * 5$	25	5^2	30	$2 * 3 * 5$
190	$2 * 5 * 19$	361	19^2	380	$2^2 * 5 * 19$
$m = 2$					
2	2	5	5	5	5
8	2^3	17	17	17	17
128	2^7	257	257	257	257
	$a = p$		$A = 2^e q$	$q = 2^{e+1} + 3$	
11	11	14	$2 * 7$	23	23
41	41	44	$2^2 * 11$	83	83
149	149	152	$2^3 * 19$	299	$13 * 23$
2141	2141	2144	$2^5 * 67$	4283	4283
107	107	110	$2 * 5 * 11$	215	$5 * 43$
881	881	884	$2^2 * 13 * 17$	1763	$41 * 43$
65	$5 * 13$	86	$2 * 43$	131	131
959	$7 * 137$	1106	$2 * 7 * 79$	1919	$19 * 101$

Table 11: super perfect numbers with translation $m = 3, 4, 6$

a	factor	A	factor	B	factor
$m = 3$					
5	5	9	3^2	12	$2^2 * 3$
$m = 4$					
2	2	7	7	7	7
4	2^2	11	11	11	11
8	2^3	19	19	19	19
32	2^5	67	67	67	67
64	2^6	131	131	131	131
2048	2^{11}	4099	4099	4099	4099
47	47	52	$2^2 * 13$	97	97
587	587	592	$2^4 * 37$	1177	$11 * 107$
341	$11 * 31$	388	$2^2 * 97$	685	$5 * 137$
$m = 6$					
4	2^2	13	13	13	13
16	2^4	37	37	37	37
49	7^2	63	$3^2 * 7$	103	103
1024	2^{10}	2053	2053	2053	2053

4 super Mersenne perfect numbers

In 2019, the prime part q of the perfect number $\alpha = 2^e q$ is studied.

If $a = 2^{e+1} - 1 + m$ is prime, then $\sigma(a) = a + 1$. Thus $\sigma(a) = a + 1 = 2^{e+1} + m$.

Letting A be $\sigma(a) - m$, we get $A = 2^{e+1}$.

From $a + 1 = 2^{e+1} + m$, it follows that

$$\sigma(A) = 2^{e+2} - 1 = 2 * 2^{e+1} - 1 = 2a - 2m + 1.$$

We obtain the following symultaneous equations.

$A = \sigma(a) - m$, $\sigma(A) = 2a - 2m + 1$.

Definition 1 *A positive integer a satisfying the above equation is said to be super Mersenne perfect number with translation m , and A a partner of a.*

Proposition 2 *If a is prime, then A is 2^e for some $e > 0$, The converse is true. (Under the hypothesis.)*

Proof.
By definition
$A = \sigma(a) + m$ and $\sigma(A) = 2a + m$.
Subtracting these

$$A - \sigma(A) = \sigma(a) - 2a.$$

If a is 2^e then $\sigma(a) - 2a = -1$. Hence,$A - \sigma(A) = -1$. Thus , A is prime.
Conversely, if A is prime, then $\sigma(A) = A + 1$. Thus $\sigma(a) - 2a = -1$. Hence, $a = 2^e$ by making use of the hypothesis.
End.

Table 12: super Mersenne perfect numbers with translation $m = -9$

a	factor	A	factor	B	factor
$m = -9$					
51	$3 * 17$	81	3^4	120	$2^3 * 3 * 5$
537	$3 * 179$	729	3^6	1092	$2^2 * 3 * 7 * 13$
4911	$3 * 1637$	6561	3^8	9840	$2^4 * 3 * 5 * 41$

Theorem 1 *If $m = -9$ and assume that $a = 3p$ for p :prime, then $A = 3^e$ for some e.*

By $m = -9$, we get $A = \sigma(a) + 9, \sigma(A) = 2a + 19$.
Using $a = 3p$ and $A = 3^e$, we get $A = \sigma(a) + 9 = 4p + 13 = 3^e$.

Proposition 3 *Assuming $4p + 13 = 3^e$, $a = 3p, (p \neq 2, 3)$ satisfies $A = \sigma(a) + 9, \sigma(A) = 2a + 19$.*

Proof.
$A = \sigma(a) + 9 = 4p + 13 = 3^e$, hence, $2\sigma(A) = 3^{e+1} - 1$.
End

If $p = \frac{3^e - 13}{4}$, then prime p is a solution.

Table 13: $m = -9$ の解 $a = 3p, A = 3^e$

e	$3 * p$
4	$51 = 3 * 17$
6	$537 = 3 * 179$
8	$4911 = 3 * 1637$
10	$44277 = 3 * 14759$
12	$398571 = 3 * 132857$
88	$3 * 242443432446880900719205485541020203890037$

Theorem 2 *When $m = -9$, assuming $a = 3p$, $A = 3^e$ is derived. (Under the next hypothesis)*

Proof.

By assumption $a = 3p$, by making use of $A = \sigma(a) + 9, \sigma(A) = 2a + 19$, we get $A = \sigma(a) + 9 = 4p + 13, \sigma(A) = 2a + 19 = 6p + 19$.

From $A - 13 = 4p, \sigma(A) - 19 = 6p$, it follows that $(12p =)3A - 39 = 2\sigma(A) - 38$.

Hence, $3A - 1 = 2\sigma(A)$.

Remark 1 (hypothesis) *For a prime p, if $(p - 1)\sigma(a) = ap - 1$, then $a = p^f$ for some $f > 0$.*

For $a < 100000$, the following are counter examples.

- $a = 7 * 11$ when $p = 5$,
- $a = 97783 = 7 * 61 * 229$ when $p = 7$,
- $a = 611 = 13 * 47$ when $p = 11$,
- $a = 1073 = 29 * 37$, $a = 2033 = 19 * 107$, when $p = 17$.

Table 14: super Mersenne perfect numbers with $m = -2, -1$

a	factor	A	factor	B	factor
$m = -2$					
5	5	8	2^3	5	5
13	13	16	2^4	9	3^2
29	29	32	2^5	17	17
61	61	64	2^6	33	$3 * 11$
509	509	512	2^9	257	257
1021	1021	1024	2^{10}	513	$3^3 * 19$
4093	4093	4096	2^{12}	2049	$3 * 683$
16381	16381	16384	2^{14}	8193	$3 * 2731$
$m = -1$					
2	2	4	2^2	3	3
14	$2 * 7$	25	5	5	5

Table 15: super Mersenne perfect numbers with translation $m = 0, 1$

a	factor	A	factor	B	factor
$m = 0$					
3	3	4	2^2	3	3
7	7	8	2^3	5	5
31	31	32	2^5	17	17
127	127	128	2^7	65	$5 * 13$
8191	8191	8192	2^{13}	4097	$17 * 241$
$m = 1$					
2	2	2	2	2	2

Table 16: super Mersenne perfect numbers with translation $m = -2$

a	factor	A	factor	B	factor
$m = 2$					
3	3	2	2	2	2
5	5	4	2^2	3	3
17	17	16	2^4	9	3^2
257	257	256	2^8	129	$3 * 43$
$m = 3$					
2	2	0	0	1	1
4	2^2	1	1	2	2
$m = 4$					
3	3	0	0	1	1
5	5	2	2	2	2
7	7	4	2^2	3	3
11	11	8	2^3	5	5
19	19	16	2^4	9	3^2
67	67	64	2^6	33	$3 * 11$
131	131	128	2^7	65	$5 * 13$
4099	4099	4096	2^{12}	2049	$3 * 683$
32771	32771	32768	2^{15}	16385	$5 * 29 * 113$

References

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